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THE

DOCTRINE

AND

APPLICATION

OF

FLUXIONS.

CONTAINING
(Beſides what is common on the Subject)
A Number of NEW IMPROVEMENTS
in the THEORY.

AND

The SOLUTION of a Variety of New, and very
Intereſting, Problems in different Branches of
the MATHEMATICKS.

PART II.

By *THOMAS SIMPSON*, F.R.S.

LONDON:
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in the *Strand*. MDCCL.

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ERRATA.

Page 286. line 9. for *Dedree*, r. *Degree*; p. 287. l. laſt, for $\dot{K}$, r. $\dot{H}$; p. 288. l. laſt, for y, r. y^{1-m}; p. 289. l. 20. for *into*, r. *by*; p. 291. l. 7. before *applicable*, r. *chiefly*; p. 293. l. 7. for *Art*. 14. r. *Art*. 252; p. 298. l. 8. for $2A^2 dx$, r. $2a^2 Ax$; p. 300. l. laſt, for $-\frac{x^9}{144a^8}$, r. $+\frac{x^9}{144a^8}$; p. 304. l. laſt but one, for n, r. p; p. 237. l. 1. for d^{2v}, r. $d^{2v}A$; p. 350. l. 2. for k, r. $2k$; l. laſt, before EG, r. EF; p. 384. l. 10. for x^{n+m-1}, r. $x^{n+m-1}\dot{x}$; and for x^{n-m-1}, r. $x^{n-m-1}\dot{x}$; p. 436. l. 23. for $\dot{z}^2$, r. $\dot{z}$; p. 465. l. 9. and 14. for $2t$ (in the Diviſor) r. $2t^5$; p. 470. l. 1. for *Problem*, r. *Propoſition*; p. 481. l. 7. dele $\times A$; p. 484. l. 5. for $\dot{x}$, r. $\dot{y}$; l. 12. for u, r. $\dot{u}$; l. 13. for uu, r. $\dot{u}\dot{u}$; l. 15. for $r-1$, read $r-m$; l. 20, for $\dot{y}y$, r. $\dot{y}\dot{y}$; p. 504. l. 18. for BT, r. BF; p. 513. l. 3. for a^2 (in the Diviſor) r. x; l. 6. for $\dot{x}$, r. $\mp\dot{x}$; alſo in l. 6. and 7. for $-$ read $+$ and for $\backsim$ r. $-$; p. 514. l. 7. and 8. for $+$, r. $-$; p. 521. l. 28. 29. and 30. for AEB, r. ABE; p. 544. l. 10. for OP^2 &c. r. AD (1) : AB $(\sqrt{1+bb})$ ::

After Line 2. p. 354. add—*when n is odd; and the Firſt and Laſt when n is even.*

38 DE 1 1 1

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THE
DOCTRINE and APPLICATION
OF
FLUXIONS.

PART the Second.

SECTION I.

The Manner of inveſtigating the FLUXIONS *of* Exponentials, *with* Thoſe *of the Sides and Angles of ſpherical Triangles.*

250. THE Method of deriving the Fluxion of any Power, x^v, of a flowing Quantity, when the Exponent (v) is given or invariable, has been already ſhewn: But, if the Exponent be variable, that Method fails; in which Caſe the Quantity x^v is call'd an Exponential; whoſe Fluxion is thus determined.

Put $z=x^v$, and let the hyperbolic Logarithm of x be denoted by y; then that of x^v (z) will, by the Nature of Logarithms, be $=vy$; and therefore its Fluxion $=\dot{v}y+v\dot{y}$: But the Fluxion of the Logarithm of z $(=x^v)$

* Art. 126. is alſo expreſſed by $\frac{\dot{z}}{z}$ *; whence we have $\frac{\dot{z}}{z}=v\dot{y}+y\dot{v}$, and conſequently $\dot{z}=zv\dot{y}+zy\dot{v}$: Which Equation, by ſubſti-
† Art. 126. tuting $\frac{\dot{x}}{x}$ for its Equal $\dot{y}$ †, becomes $\dot{z}=zy\dot{v}+\frac{zv\dot{x}}{x}=$ $x^{v}y\dot{v}+x^{v}\times\frac{v\dot{x}}{x}=x^{v}y\dot{v}+vx^{v-1}\dot{x}=x^{v}\dot{v}\times\overline{hyp.\ Log.\ x+vx^{v-1}\dot{x}}$.

The ſame otherwiſe, without introducing the Properties of Logarithms.

251. Let $1+z=x$, and $n+w=v$, ſuppoſing n conſtant and w variable: Then $x^{v}=\overline{1+z}|^{n+w}=\overline{1+z}|^{n}\times\overline{1+z}|^{w}=\overline{1+z}|^{n}\times\overline{1+wz+\frac{w}{1}\times\frac{w-1}{2}\times z^{2}+\frac{w}{1}\times\frac{w-1}{2}\times\frac{w-2}{3}\times z^{3}+}$ &c.* $=\overline{1+z}|^{n}\times$
* Art. 99. $\overline{1+wz+\overline{\tfrac{1}{2}w^{2}-\tfrac{1}{2}w}\times z^{2}+\overline{\tfrac{1}{6}w^{3}-\tfrac{1}{2}w^{2}+\tfrac{1}{3}w}\times z^{3}+}$ &c. whoſe Fluxion, found the common Way, is $n\dot{z}\times\overline{1+z}|^{n-1}\times\overline{1+wz+\overline{\tfrac{1}{2}w^{2}-\tfrac{1}{2}w}\times z^{2}+\overline{\tfrac{1}{6}w^{3}-\tfrac{1}{2}w^{2}+\tfrac{1}{3}w}\times z^{3}}$ &c. $+\overline{1+z}|^{n}\times\overline{\dot{w}z+w\dot{z}+\overline{w\dot{w}-\tfrac{1}{2}\dot{w}}\times z^{2}+\overline{\tfrac{1}{2}w^{2}-\tfrac{1}{2}w}\times 2z\dot{z}+\overline{\tfrac{1}{2}w^{2}\dot{w}-w\dot{w}+\tfrac{1}{3}\dot{w}}\times z^{3}+\overline{\tfrac{1}{6}w^{3}-\tfrac{1}{2}w^{2}+\tfrac{1}{3}w}\times 3z^{2}\dot{z}}$ &c. which, by ſubſtituting $\dot{x}$ and $\dot{v}$ for their Equals $\dot{z}$ and $\dot{w}$, becomes $n\dot{x}\times\overline{1+z}|^{n-1}\times\overline{1+wz+\overline{\tfrac{1}{2}w^{2}-\tfrac{1}{2}w}\times z^{2}+}$ &c. $+\overline{1+z}|^{n}\times\overline{\dot{v}z+w\dot{x}+\overline{w\dot{v}-\tfrac{1}{2}\dot{v}}\times z^{2}+}$ &c.

But, if w be, now, ſuppoſed to vaniſh, we ſhall have the true Value of the Fluxion when $v=n$; which, in that Circumſtance, appears to be $=n\dot{x}\times\overline{1+z}|^{n-1}$

$+\overline{1+z}^{v}\times\overline{z\dot v-\frac{1}{2}z^2\dot v+\frac{1}{3}z^3\dot v-\frac{1}{4}z^4\dot v}$ &c. $=v\dot x\times x^{v-1}$ $+\dot v x^{v}\times\overline{z-\frac{1}{2}z^2+\frac{1}{3}z^3-\frac{1}{4}z^4}$ &c. *Q. E. I.*

It is plain, becauſe the Series, $z-\frac{1}{2}z^2+\frac{1}{3}z^3$ &c. here brought out, is known to expreſs the Fluent of $\frac{\dot z}{1+z}$, or the hyperbolic Logarithm of $1+z$ *, that the two Concluſions agree exactly with each other: From either of which the following *Rule*, for the Fluxions of Exponentials, is deduced.

* Art. 126.

252. *To the Fluxion found by the common Rule* (Art. 14.) *conſidering the Exponent as conſtant, add the Quantity ariſing by multiplying the Fluxion of the Exponent, the hyperbolic Logarithm of the Root, and the propoſed Quantity itſelf, continually, together: The Sum will be the Fluxion when the Exponent is variable.*

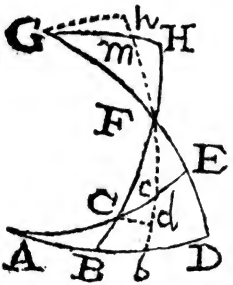
Thus, for Example, let the Quantity propoſed be $\overline{a^2+z^2}^{z}$, then the Fluxion thereof will be $z\times 2z\dot z\times\overline{a^2+z^2}^{z-1}+\dot z\times\overline{a^2+z^2}^{z}\times$ *hyp. Log.* $\overline{a^2+z^2}$.

But, if the Root is conſtant, and only the Exponent variable, the *Exponential* will be more ſimple; and its Fluxion will then be had by *barely multiplying the Quantity itſelf by the Product under the Logarithm of the Root and the Fluxion of the Exponent.*

Thus, the Fluxion of a^x will be expreſſed by $a^x\times\dot x$ $\times$ *hyp. Log.* a; and that of $\overline{a^2+b^2}^{nx}$ by $\overline{a^2+b^2}^{nx}\times n\dot x$ $\times$ *hyp. Log.* $\overline{a^2+b^2}$. Theſe Kind of *Exponentials* oftner occur, in Practice, than any other; but, as it is very rare that we meet with *any*, I ſhall therefore proceed now to the other Conſideration propoſed in the Head of this Section; namely, the Method of determining the Fluxions of the Sides and Angles of ſpherical Triangles (a Thing very uſeful in practical Aſtronomy) which I ſhall deliver in the following Propoſitions.

PROPOSITION I.

253. *To determine the Ratio of the Fluxions of the ſeveral Parts of a right-angled ſpherical Triangle; ſuppoſing the Hypothenuſe, one Leg, or one Angle, to remain conſtant, while the other Parts vary.*



Let A, F, and G be the Poles of the three Great-Circles DEFG, ABD, and ACE; whereof the Poſition of each is ſuppoſed to continue invariable, while another Great-Circle HFCB is conceived to revolve about the Pole F: Whence, if GH be ſuppoſed perpendicular to FH, three variable right-angled Triangles, FGH, FCE, and ABC, will be formed; in the firſt whereof, the Hypothenuſe FG will remain conſtant; in the ſecond, the Leg EF; and in the third the Angle A.

Let B*b* (*q*) be the Fluxion (or indefinitely ſmall Increment *) of the Baſe AB, or the Angle F; and let C*d* meet the Great-Circle *b*F*b*, at Right-angles, in *d*; then it will be (*per Spherics*) as *Sin.* FB (*Rad.*) : *Sin.* FC :: B*b* (*q*) : C*d* $= \frac{\textit{Sin.}\ \mathrm{FC}}{\textit{Rad.}} \times q = \frac{\textit{Co-ſ.}\ \mathrm{BC}}{\textit{Rad.}} \times q$;

And, *Tang.* C : *Rad.* :: C*d* $\left(\frac{\textit{Co-ſ.}\ \mathrm{BC}}{\textit{Rad.}} \times q\right) : \frac{\textit{Co-ſ.}\ \mathrm{BC}}{\textit{Tang.}\ \mathrm{C}} \times q =$ the Fluxion of BC.

Moreover, *Sin.* C : *Rad.* :: C*d* $\left(\frac{\textit{Co-ſ.}\ \mathrm{BC}}{\textit{Rad.}} \times q\right) : \frac{\textit{Co-ſ.}\ \mathrm{BC}}{\textit{Sin.}\ \mathrm{C}} \times q =$ the Fluxion of AC.

* Art. 134.

Laſtly, *Sine* of FB *(Rad.)* : *Sin.* FH (BC) :: Bb (q) : $\frac{\textit{Sin.}\ \mathrm{BC}}{\textit{Rad.}} \times q$ (= Hm) = the Fluxion of GH, or its Complement C.

Now, if the ſeveral Quantities, in theſe three Equations for the Triangle ABC, be expounded by their reſpective Equals in the other two Triangles CEF and FGH, we ſhall alſo have

$$\frac{\textit{Sin.}\ \mathrm{CF}}{\textit{Tang.}\ \mathrm{C}} \times q = -\ \textit{Flux.}\ \mathrm{CF}.$$

$$\frac{\textit{Sin.}\ \mathrm{CF}}{\textit{Sin.}\ \mathrm{C}} \times q = -\ \mathrm{Flux.\ CE}.$$

$$\frac{\textit{Co-ſ.}\ \mathrm{CF}}{\textit{Rad.}} \times q = \mathrm{Flux.\ C}.$$

And

$$\frac{\textit{Co-ſ.}\ \mathrm{FH}}{\textit{Co-tang.}\ \mathrm{GH}} \times q = \mathrm{Flux.\ FH}.$$

$$\frac{\textit{Co-ſ.}\ \mathrm{FH}}{\textit{Co-ſ.}\ \mathrm{GH}} \times q = \mathrm{Flux.\ G}.$$

$$\frac{\textit{Sin.}\ \mathrm{FH}}{\textit{Rad.}} \times q = -\ \mathrm{Flux.\ GH}.$$

Q. E. I.

COROLLARY I.

254. Hence, if, in any right angled Spherical-Triangle, the Hypothenuſe be denoted by h, the two Legs by L and l, the Angles, reſpectively, adjacent to them by A and a, we ſhall, by ſubſtituting above, have three Equations for each of the three Caſes. From the Compariſon and Compoſition of which, the three following Tables are deduced; exhibiting all the different Varieties that can poſſibly happen, whether an Angle, a Leg, or the Hypothenuſe be ſuppoſed invariable.

TABLE I.

When one Angle *A* is invariable,

$$\dot{L}=\frac{\text{Tang. }a}{\text{Co-ſ. }l}\times\dot{l}=\frac{\text{Sin. }a}{\text{Co-ſ. }l}\times\dot{h}=\frac{\text{Rad.}}{\text{Sin. }l}\times\dot{a}$$

$$\dot{l}=\frac{\text{Co-ſ. }l}{\text{Tang. }a}\times\dot{L}=\frac{\text{Co ſ. }a}{R}\times\dot{h}=\frac{\text{Co·tang. }l}{\text{Tang. }a}\times\dot{a}$$

$$\dot{h}=\frac{\text{Co-ſ. }l}{\text{Sin. }a}\times\dot{L}=\frac{R}{\text{Co-ſ. }a}\times\dot{l}=-\frac{\text{Co-tang. }l}{\text{Sin. }a}\times\dot{a}$$

$$\dot{a}=\frac{\text{Sin. }l}{R}\times\dot{L}=\frac{\text{Tang. }a}{\text{Co-tang. }l}\times\dot{l}=\frac{\text{Sin. }a}{\text{Co-tang. }l}\times\dot{h}$$

TABLE II.

When one Leg *L* is invariable,

$$\dot{A}=\frac{\text{Tang. }a}{\text{Sin. }h}\times\dot{h}=\frac{\text{Sin. }a}{\text{Sin. }h}\times\dot{l}=-\frac{R.}{\text{Co-ſ. }h}\times\dot{a}$$

$$\dot{a}=-\frac{\text{Co-ſ. }h}{R}\times\dot{A}=-\frac{\text{Sin. }a}{\text{Tang. }h}\times\dot{l}=-\frac{\text{Tang. }a}{\text{Tang. }h}\times\dot{h}$$

$$\dot{h}=\frac{\text{Sin. }h}{\text{Tang. }a}\times\dot{A}=\frac{\text{Co-ſ. }a}{R}\times\dot{l}=-\frac{\text{Tang. }h}{\text{Tang. }a}\times\dot{a}$$

$$\dot{l}=\frac{\text{Sin. }h}{\text{Sin. }a}\times\dot{A}=\frac{R.}{\text{Co-ſ. }a}\times\dot{h}=-\frac{\text{Tang. }h}{\text{Sin. }a}\times\dot{a}$$

TABLE III.

When the Hyp. is invariable,

$$\dot{A}=-\frac{\text{Co·tang. }l}{\text{Co-ſ. }L}\times\dot{L}=-\frac{\text{Co-ſ. }l}{\text{Co·ſ. }L}\times\dot{a}=\frac{R}{\text{Sin. }L}\times\dot{l}$$

$$\dot{L}=-\frac{\text{Co-ſ. }L}{\text{Co·tang. }l}\times\dot{A}=\frac{\text{Sin. }l.}{R.}\times\dot{a}=-\frac{\text{Tang. }l}{\text{Tang. }L}\times\dot{l}$$

Where, and alſo in the two preceding Tables, the Leg *L* is adjacent to the Angle *A*, and the Leg *l* to the Angle *a*.

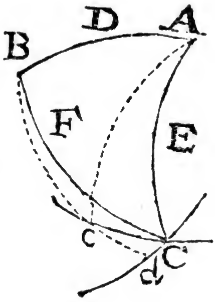
COROLLARY II.

255. From the third original Equation, expreſſing the Fluxion of the Angle *C* (*Vid. Art.* 253.) it appears that the Superficies of any Spherical-Triangle ABC, is proportional to the Exceſs of its three Angles above two Right-Angles. For (BC*db*) the Fluxion of the *Triangle* ABC, is $= \textit{Sine}\ BC \times Bb$, by *Art.* 161.) which being to, $\frac{\textit{Sin.}\ BC}{\textit{Rad.}} \times Bb$, the Fluxion of the Angle *C*, above ſpecified, in the conſtant Ratio of Radius to Unity, the Fluents themſelves (properly corrected) muſt therefore be in that Ratio; that is, the Superficies of the Triangle ABC will always be proportional to the Increaſe of the Angle *C*, from its coinciding with *A*, or as the Exceſs of *A* and *C* above two Right-Angles.

PROPOSITION II.

256. *To determine the Ratio of the Fluxions, or the indefinitely ſmall Increments, of the different Parts of an oblique Spherical-Triangle* ABC; *two Sides thereof* AB, AC *being invariable, in Length.*

Let C*c* be an indefinitely ſmall Part of the Parallel deſcribed by the Extreme C of the given Side AC, in its Motion about the given Point A; moreover, let C*d* be Part of another Parallel, whoſe Pole is the given Point B; let the Great-Circle B*c* meet C*d* in *d*; and let the three Sides, AB, AC, and BC, of the Triangle be denoted by *D*, *E*, and *F* reſpectively.



Then (*per Spherics*) we ſhall have

$$R : S.E :: \text{CA}c\ (\dot{A}) : \text{C}c = \frac{S.E.}{R} \times \dot{A};$$

And, $R : S.F :: \text{CB}d\ (\dot{B}) : \text{C}d = \frac{S.F}{R} \times \dot{B}.$

Alſo, $R : S.\ d\text{C}c\ (\text{ACB}) :: \text{C}c : \dot{F} = \frac{S.E \times S.C}{R^2} \times \dot{A}:$ But $S.C : S.D :: S.B : S.E$; therefore $S.E \times S.C = S.D \times S.B$, and conſequently $\dot{F}$, alſo, $= \frac{S.D \times S.B}{R^2} \times \dot{A}.$

Again, $R : \text{Co-ſ.}\ d\text{C}c\ (\text{ACB}) :: \text{C}c \left(\frac{S.E}{R} \times \dot{A}\right) : \frac{S.E. \times \text{Co-ſ.}\ C}{R^2} \times \dot{A}\ (= \text{C}d) = \frac{S.F}{R} \times \dot{B};$

Whence $\dot{B} = \frac{S.E \times \text{Co-ſ.}\ C}{R \times S.F} \times \dot{A}.$

Laſtly, $\text{Co t.}\ c\text{C}d\ (C) : R :: \text{C}d \left(\frac{S.F}{R} \times \dot{B}\right) : \dot{F} = \frac{S.F}{\text{Co. t.}\ C} \times \dot{B}.$

Whence, by the very ſame Argument (ſubſtituting D for E, and C for B in the two laſt Equations) we likewiſe have $\dot{C} = \frac{S.D \times \text{Co-ſ.}\ B}{R \times S.F} \times \dot{A}$, and $\dot{F}$ $\left(= \frac{S.F}{\text{Co-t.}\ C} \times \dot{B}\right) = \frac{S.F}{\text{Co-t.}\ B} \times \dot{C}.$

Now, from the Equations thus found, it is manifeſt,

1°. $\dot{A} : \dot{F} :: R^2 : S.D \times S.B$ $(:: \text{Co-ſeca.}\ D : S.B)$

2°. $\dot{A} : \dot{B} :: R \times S.F : S.E \times \text{Co-ſ.}\ C$

3°. $\dot{A} : \dot{C} :: R \times S.F :: S.D \times \text{Co-ſ.}\ B$

4°. $\dot{B} : \dot{F} :: \text{Co-t.}\ C : S.F$

5°. $\dot{C} : \dot{F} :: \text{Co-t.}\ B : S.F$

6°. $\dot{B} : \dot{C} :: \text{Co-t.}\ C : \text{Co-t.}\ B$ $(:: T.B : T.C)$ Q. E. I.

257. Theſe Proportions, for the Fluxions of the Parts of a Spherical-Triangle, are very uſeful in various Caſes in *Practical Aſtronomy*; whereof I ſhall here put down one or two Inſtances.

The firſt is; To determine the annual Alteration of the Declination and Right-Aſcenſion of a fixt Star, through the Preceſſion of the Equinox.

Here *A* muſt denote the Pole of the Ecliptic, *B* that of the Equinoctial, and *C* the Place of the Star; and then (by the firſt and fourth Proportions) we have

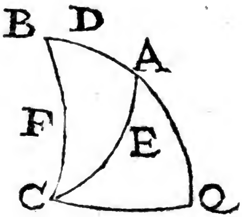
$$Co\text{-}ſeca.\ D : Sin.\ B :: \dot{A} : \dot{F};\ \text{and}$$

$$S.\ F : Co\text{-}t.\ C :: \dot{F} : \dot{B};$$

That is, 1°, As the Co-ſecant of the Obliquity of the Ecliptic is to the Sine of the Star's Right-Aſcenſion from the *ſolſtitial Colure*, ſo is the *Preceſſion* of the Equinox, or Alteration of Longitude, to the Alteration of Declination.

2°. As the Co-ſine of the Star's Declination is to the Co-tangent of its Angle of Poſition, ſo is the Alteration of Declination (found as above) to the Alteration of Right-Aſcenſion correſponding.

The ſecond Example is to find how much the Amplitude, and the Time of the apparent Riſing and Setting of the Sun, or a Star, are affected by Refraction.



In this Caſe *A* muſt denote the Pole of the Equator, and *B* the *Zenith*, and the Side BC muſt be an Arch of 90 Degrees, ſo that the Star *C* may coincide with the Horizon QC: Then, from the very ſame Proportion, we have,

$$Sin.\ B : Co\text{-}ſeca.\ D :: \dot{F} : \dot{A},$$

$$\text{And, } R : Co\text{-}t.\ C :: \dot{F} : \dot{B}$$

$$\text{But, } R : Co\text{-}t.\ C\ (T.\ QCA) :: Sin.\ B\ (CQ) : Co\text{-}tang.\ D\ (Tang.\ QA)$$

Hence it appears,

1°. That, as the Co-ſine of the true Amplitude (conſidered independent of Refraction) is to the Tangent of the Pole's Elevation, ſo is the given horizontal Refraction to the Difference of Amplitudes thence ariſing.

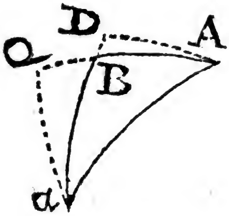
2°. And, that, as the Co-ſine of the true Amplitude is to the Secant of the Pole's Elevation, ſo is the ſaid horizontal Refraction to the Effect thereof in the Time of Riſing, or Setting of the Sun, or Star.

But this laſt Proportion may be otherwiſe expreſſed, without the Amplitude: Thus,

$S.AB \times S.AC \times S.A : R^3 ::$ the horizontal Refraction, to the ſame Effect.

PROPOSITION III.

258. *To determine the ſame as in the preceding Problem; ſuppoſing one Side* AB *and one of its adjacent Angles,* B, *to continue invariable.*



If from the End of the given Side, oppoſite to the given Angle, a Perpendicular AD be let fall, that Perpendicular, as well as the Segment BD cut off thereby, will be a conſtant Quantity, while the other Parts of the Triangle A*a*D vary, by the Motion of *a* along the Arch *a*BD. Therefore the Problem is reſolved by *Caſe* 2. of right-angled Triangles. *Vid. Art.* 254.

259. It may not be amiſs to give one Example of the Uſe of this laſt Propoſition: Which ſhall be, in finding the Parallax of a Planet in Longitude and Latitude; that of Altitude being given.

Here *A* muſt ſtand for the Pole of the Ecliptic, *B* the Zenith, and *a* the Planet: Then, if the Hypothenuſe A*a* be denoted by h, the Leg. D*a* by l, and the given Parallax, in Altitude, by $\dot{l}$, it will appear, from the Place above quoted, that $\dot{A}$ (the Parallax in Long.) will be $= \frac{Sin.\ a}{Sin.\ b} \times \dot{l} = \frac{Sin.\ BaA}{Sin.\ Aa} \times \dot{l}$, and $\dot{b}$ (the Parallax in Lat.) $= \frac{\text{Co-ſ.}\ a}{Rad.} \times \dot{l} = \frac{\text{Co-ſ.}\ BaA}{Rad.} \times \dot{l}$.

If the Planet be in (or very near) the Ecliptic, and *a*Q be ſuppoſed a Portion of the Ecliptic, meeting AB, at Right-Angles, in Q, then *(per Spherics)* $\frac{Sin.\ BaA}{Sin.\ Aa} \left(\frac{\text{Co-ſ.}\ BaQ}{Radius}\right) = \frac{Tang.\ Qa}{Tang.\ Ba}$; alſo $\frac{\text{Co-ſ.}\ BaA}{Rad.} \left(\frac{Sin.\ BaQ}{Rad.}\right) = \frac{Sin.\ QB}{Sin.\ Ba}$; whence, by ſubſtituting theſe Values above, we ſhall, *in this Caſe*, have $\dot{A} = \frac{Tang.\ Qa}{Tang.\ Ba} \times \dot{l}$ and $\dot{b} = \frac{Sin.\ QB}{Sin.\ Ba} \times \dot{l}$; that is, in Words,

As, the Tangent of the Planet's Zenith Diſtance, is to the Tangent of its Longitude from the nonageſimal Degree of the Ecliptic, ſo is the Parallax in Altitude to the Parallax in Longitude.

And, as the Sine of the Zenith Diſtance to the Coſine of the Altitude of the nonageſimal Degree, ſo is the Parallax in Altitude to the Parallax in Latitude.

Becauſe the Parallax in Altitude, the horizontal Parallax *(M)* being given, is nearly $= \frac{Sin.\ Ba}{Rad.} \times M$, if this Value be ſubſtituted for $\dot{l}$, in the two laſt Equations, we ſhall get $\dot{b} = \frac{Sin.\ QB}{Rad.} \times M$, and $\dot{A} = \frac{Tang.\ Qa \times Sin.\ Ba}{Rad. \times Tang.\ Ba} \times M = \frac{Sin.\ AB \times Sin.\ BAa}{Rad.^2} \times M$.

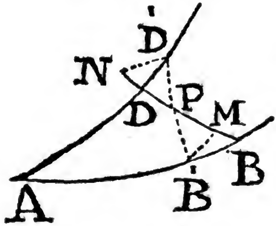
Whence, we have theſe two other Theorems, for finding the required Parallaxes immediately from the horizontal Parallax, without either the Altitude or its *Parallax*.

1. As Radius to the Co-ſine of the Altitude of the *nonageſimal Degree* of the Ecliptic, ſo is the horizontal Parallax to the Parallax in Latitude.

2. And as the Square of Radius to the Rectangle under the Sines of the Altitude of the *nonageſimal Dedree* and the Planet's Longitude from thence, ſo is the horizontal Parallax to the Parallax in Longitude.

PROPOSITION IV.

260. *Still, to determine the ſame Thing; ſuppoſing, one Angle* A, *and the Length of its oppoſite Side* BD *(or* B́D́*) to remain conſtant.*



Let B́D́ (equal to BD) interſect BD in an indefinitely ſmall Angle at *P*, and meet AB and AD in *B́* and *D́*; alſo in BD produced let there be taken PN = PD́ and PM = PB́, and let *N*, *D́*, and *M*, *B́* be joined.

Since, by Hypotheſis, DB=D́B́=MN, if from the firſt and laſt of theſe equal Qnantities DM, common, be taken away, there will remain BM=DN.

Moreover, ſince the Triangles BMB́ and DND́, in their ultimate State, may be conſidered as rectilineal, and right-angled at *M* and *N* *, it will therefore be, as

BM : BB́ :: *Co-ſ. B* : *Radius*

And DN : DD́ :: *Co-ſ. D* :: *Radius.*

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* Art. 134.

From whence, the Extremes in both Proportions being the ſame, we have $B\dot{B} : D\dot{D} :: \textit{Co-ſ.}\ D : \textit{Co-ſ.}\ B$: And therefore, if AB be denoted by *H* and AD by *K*, it appears that $\dot{H} : \dot{K} :: \textit{Co-ſ.}\ D : \textit{Co-ſ.}\ B$.

Again, *per Spherics*, *Sin. A* : *Sin.* BD (*G*) :: *Sin. D* : *Sin. H* :: *Flux. Sin. D* : *Flux. Sin. H*; becauſe, the Sines themſelves being in a conſtant Ratio, their Fluxions muſt be in the ſame Ratio: But the Fluxion of the Sine of any Arc, or Angle, is to the Fluxion of the Arc or Angle itſelf, as the Co-ſine to Radius*: Therefore the *Flux. Sin.* D being $= \frac{\textit{Co-ſ.}\ D}{\textit{Rad.}} \times \dot{D}$, and *Flux. Sin.* $H = \frac{\textit{Co-ſ.}\ H}{\textit{Rad.}} \times \dot{H}$, it follows that, $\textit{Sin.}\ A : \textit{Sin.}\ G :: \textit{Co-ſ.}\ D \times \dot{D} : \textit{Co-ſ.}\ H \times \dot{H}$; or $\dot{D} : \dot{H} :: \textit{Sin.}\ A \times \textit{Co-ſ.}\ H : \textit{Sin.}\ G \times \textit{Co-ſ.}\ D$: And, by the very ſame Argument, $\dot{B} : \dot{K} :: \textit{Sin.}\ A \times \textit{Co.ſ.}\ K : \textit{Sin.}\ G \times \textit{Co-ſ.}\ B$. Now, by compounding the former of theſe two Proportions with the firſt above given, we get, $\dot{D} : \dot{K} :: \textit{Sin.}\ A \times \textit{Co-ſ.}\ H : \textit{Sin.}\ G \times \textit{Co-ſ.}\ B$. And, by compounding this laſt with $\dot{K} : \dot{B} :: \textit{Sin.}\ G \times \textit{Co-ſ.}\ B : \textit{Sin.}\ A \times \textit{Co-ſ.}\ K$ (that immediately preceding it) we alſo obtain $\dot{D} : \dot{B} :: \textit{Co-ſ.}\ H : \textit{Co-ſ.}\ K$.

Whence, by collecting theſe ſeveral Proportions together, we have the following Table, for all the different Caſes.

$\dot{H} : \dot{K} :: \textit{Co-ſ.}\ D : \textit{Co-ſ.}\ B$

$\dot{D} : \dot{B} :: \textit{Co-ſ.}\ H : \textit{Co-ſ.}\ K$

$\dot{D} : \dot{H} :: \textit{Tang.}\ D : \textit{Tang.}\ H$

$\dot{B} : \dot{K} :: \textit{Tang.}\ B : \textit{Tang.}\ K$

$\dot{K} : \dot{D} :: \textit{Sin.}\ G \times \textit{Co-ſ.}\ B : \textit{Sin.}\ A \times \textit{Co-ſ.}\ H$

$\dot{K} : \dot{B} :: \textit{Sin.}\ G \times \textit{Co-ſ.}\ D : \textit{Sin.}\ A \times \textit{Co-ſ.}\ K$

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* Art. 142.

It may be obſerved, that the fourth and the laſt are no new Caſes, but only the third and fifth repeated: And that, though the former of the two, laſt named, differs from that found above; yet it is very eaſily deduced from it: For, ſince it appears that $\dot{D} : \dot{H} :: \frac{Sin.\ A.}{\text{Co-ſ.}\ D} : \frac{Sin.\ G.}{\text{Co-ſ.}\ H}$, and becauſe $Sin.\ A : Sin.\ G :: Sin.\ D : Sin.\ H$, it follows that $\dot{D} : \dot{H} :: \frac{Sin.\ D}{\text{Co-ſ.}\ D} : \frac{Sin.\ H}{\text{Co-ſ.}\ H} :: Tang.\ D : Tang.\ H.$ Q. E. I.

There is yet another Problem, when two Angles remain conſtant; but this, by taking the Triangle formed by the Poles of the three given Circles, is reduced to *Problem* 2.

SECTION II.

Of the Reſolution of fluxional Equations, or the Manner of finding the Relation of the flowing Quantities from that of the Fluxions.

261. WHEN an Equation, expreſſing the Relation of the Fluxions of the two variable Quantities, contains *only* one of thoſe Fluxions with its reſpective flowing Quantity in each Term, the Relation of the Quantities will be obtained by finding the Fluent of every Term; as has been already taught, in *Sect.* VI. *Part* I.

Thus, if $ax^2\dot{x} = y^3\dot{y}$, then will $\frac{ax^3}{3} = \frac{y^4}{4}$.

And, if $x^n y^m \dot{x} = a\dot{y}$; by reducing it firſt to $x^n \dot{x} = ay^{-m}\dot{y}$ (ſo that its variable Quantities may be ſeparated) we have $\frac{x^{n+1}}{n+1} = \frac{ay}{1-m}$.

But,

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But, if the given Equation has its indeterminate Quantities and their Fluxions ſo complicated together, that it cannot be brought under the Form there preſcribed, the Task will become much more difficult; nor is there any general Method to be given for ſuch Kinds of Equations, whereof there are an infinite Variety.

The Method of Infinite Serieſes (in ſome meaſure explained already, and more fully conſidered hereafter) is indeed very comprehenſive, and may be apply'd to good Purpoſe in various Caſes; but, being tedious and attended with a Number of Inconveniencies, it is a Method we ought never to have Recourſe to till we have try'd what may be, otherways, effected, by help of ſuch particular Rules and Obſervations as we have been able to collect.

Accordingly, I ſhall, here, firſt point out ſome of the moſt proper Ways to be tried, in order, if poſſible, to bring out the Solution without an Infinite Series.

262. *The firſt Method is, by multiplying, or dividing, the given Equation into ſome Power or Product of the Quantities concerned; ſo as to bring it, if poſſible, under the Form of ſuch Fluxions, as, we know, do ariſe, if not from the firſt, yet from the ſecond, or third, of the three general Rules in the direct Method.*

Thus, if the given Equation be $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \frac{x^m \dot{x}}{ay^n}$; then, the whole being multiply'd by xy, ſo that the two firſt Terms, $y\dot{x} + x\dot{y}$, may become the (known) Fluxion of the Rectangle xy *, there ariſes $y\dot{x} + x\dot{y} = \frac{x^{m+1}\dot{x}}{ay^{n-1}}$: But ſtill we are at a Loſs for the Fluent of the laſt Term, unleſs n be taken $= 1$ (ſo that y may vaniſh.) In that Caſe we have $xy = \frac{x^{m+2}}{\overline{m+2} \times a}$; expreſſing the Relation of the Fluents when that of the Fluxions is $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \frac{x^m \dot{x}}{ay}$: Which appears to be the only Caſe, of the given Equation, where this Method is of Uſe.

* Art. 1[illegible].

Again, let the Equation $\frac{p\dot x}{x} + \frac{r\dot y}{y} = \frac{x^m \dot x}{ay^n}$ be propoſed.

Here, multiplying by $x^p y^r$ (where the Exponents are the ſame as the Coefficients of $\frac{\dot x}{x}$ and $\frac{\dot y}{y}$) we get $px^{p-1}\dot x \times y^r + x^p \times ry^{r-1}\dot y = \frac{x^{m+p}\dot x}{ay^{n-r}}$; in which the former Part of the Equation is known to expreſs the Fluxion of $x^p y^r$ *. Therefore, when $n = r$, the Relation of the Fluents may be found, and will be expreſſed by $x^p y^r = \frac{x^{m+p+1}}{\overline{m+p+1} \times a}$: Which, if no Correction by a conſtant Quantity be neceſſary, may be reduced to $y^r = \frac{x^{m+1}}{\overline{m+p+1} \times a}$.

* Art. 15.

The ſame Method may alſo be extended to Fluxions of the higher Orders: Let $\ddot x - x\dot z^2 = f\dot z^2$ (which Equation occurs hereafter, in the Reſolution of a Problem of ſome Difficulty.) Then, multiplying by $\dot x$, it becomes $\dot x\ddot x - x\dot x\dot z^2 = f\dot z^2\dot x$; where, $\dot z$ being conſtant, each Term admits, now, of a perfect Fluent, and we therefore have $\frac{\dot x^2}{2} - \frac{x^2\dot z^2}{2} = fx\dot z^2$: From whence, ſuppoſing no Correction neceſſary, $\dot z = \frac{\dot x}{\sqrt{2fx + xx}}$, and $z =$ *hyp. Log.* $f + x + \sqrt{2fx + x^2}$ (*by Art.* 126.)

263. *It may happen that the Solution of an Equation will become more eaſy by firſt taking the Fluxion thereof; when, by that means, ſome of the Terms deſtroy each other.*

The following is an Inſtance of it (which, alſo, occurs hereafter) Let $y + \frac{\dot y \times \overline{a - x}}{\dot x} = x - \frac{y\dot x}{\dot y}$: Whoſe Fluxion, making $\dot{x}$ conſtant, is $\dot{y}+\frac{\ddot{y}\times\overline{a-x}-\dot{x}\dot{y}}{\dot{x}}=\dot{x}-\frac{\dot{y}\dot{x}\dot{y}-y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$: Which, by reaſon of the Terms deſtroying one another, is reduced to $\frac{\ddot{y}\times\overline{a-x}}{\dot{x}}=\frac{y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$: Therefore, by expunging $\ddot{y}$, *&c.* we get $\dot{y}y^{-\frac{1}{2}}=\dot{x}\times\overline{a-x}|^{-\frac{1}{2}}$, and conſequently $2y^{\frac{1}{2}}=-2\times\overline{a-x}|^{\frac{1}{2}}+$ *ſome conſtant Quantity.*

264. *Another Method, applicable to Equations, of the firſt Order of Fluxions, wherein only one of the two variable Quantities (x or y) enters, is, to ſubſtitute for the Ratio of the two Fluxions ($\dot{x}$ and $\dot{y}$): From whence the Value of that Quantity will be had, immediately, in Terms of the ſaid aſſumed Ratio: And then, by taking its Fluxion, that of the other Quantity (and from thence the Quantity itſelf) will become known.*

Thus, let $a\dot{x}\dot{y}^3=y\times\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^2$ (being the Equation of the Curve that generates the Solid of the *leaſt Reſiſtance*, when the Bulk and greateſt Diameter are given.) Then, by putting $\frac{\dot{x}}{\dot{y}}=v$, and ſubſtituting above, we get $av\dot{y}^4=y\times\overline{v^2\dot{y}^2+\dot{y}^2}|^2=y\dot{y}^4\times\overline{v^2+1}|^2$; and conſequently $y=\frac{av}{\overline{v^2+1}|^2}$: Therefore $\dot{y}=\frac{a\dot{v}-3av^2\dot{v}}{\overline{vv+1}|^3}$; and conſequently $\dot{x}$ $(=v\dot{y})=\frac{av\dot{v}-3av^3\dot{v}}{\overline{vv+1}|^3}$: Whoſe Fluent may be found, from *Art.* 84. or, otherwiſe, thus: Put $w^2=v^2+1$; then $v^2=w^2-1$, and $w\dot{w}=v\dot{v}$; by ſubſtituting which Values there ariſes $\dot{x}=\frac{aw\dot{w}-3aw\dot{w}\times\overline{w^2-1}}{w^6}=4a\dot{w}w^{-5}-3a\dot{w}w^{-3}$; and

therefore $x = \frac{4aw^{-4}}{-4} - \frac{3aw^{-2}}{-2} = -\frac{a}{w^4} + \frac{3a}{2w^2} = \frac{3aw^2-2a}{2w^4}$ $= \frac{3a\times\overline{v^2+1}-2a}{2\times\overline{vv+1}\vert^2} = \frac{a\times\overline{3vv+1}}{2\times\overline{vv+1}\vert^2}$; which, corrected (by taking y, or $v=0$) becomes $x = \frac{a\times\overline{3vv+1}}{2\times\overline{vv+1}\vert^2} - \frac{a}{2}$. From this Equation, by completing the Square, &c. v may be found in Terms of x; whence the correſponding Value of y $\left(= \frac{av}{\overline{vv+1}\vert^2}\right)$ will alſo be known.

265. *The fourth Method*, which chiefly obtains when one of the indeterminate Quantities and its Fluxion, ariſe but to a ſingle Dimenſion each, *may be thus:*

Let the Value of that Quantity, which is leaſt involved, be firſt ſought, from the fictitious Equation ariſing by neglecting all the Terms in the given Equation, where neither that Quantity, nor its Fluxion, are found: Then, to that Value, let ſome Power, or Powers, of the other Quantity, with unknown Coefficients, be added (according to the Dimenſions of the Terms neglected) and let the Sum be ſubſtituted in the given Equation, as the true Value of the firſt mentioned Quantity: By which means a new Equation will reſult; from whence the aſſumed Coefficients may, ſometimes, be determined.

Ex. Let the given Equation be $cx^2\dot{x}+y\dot{x}=a\dot{y}$.

By neglecting $cx^2\dot{x}$, or feigning $y\dot{x}=a\dot{y}$, we get $\frac{\dot{x}}{a}=\frac{\dot{y}}{y}$: and conſequently $\frac{x}{a}=$ hyp. Log. y — hyp. Log. d * $=$ hyp. Log. $\frac{y}{d}$; d being any conſtant Quantity, which the Nature of the Problem may require. Hence $\frac{y}{d}=$ the Number whoſe hyperbolical Logarithm is $\frac{x}{a}$: Which Number, if M be put for (2,71828 &c.)

* Art. 126. and 78.

the Number whoſe hyp. Log. is Unity, will be expreſſed by $\overline{M}|^{\frac{x}{a}}$ (ſince it is evident that the hyp. Log. hereof is $\frac{x}{a} \times$ Log. $M = \frac{x}{a}$) : Therefore $\frac{y}{d} = \overline{M}|^{\frac{x}{a}}$ and $y = d \times \overline{M}|^{\frac{x}{a}}$. Now, to the Value thus found, let there be added $Ax^2 + Bx + C$, in order to get the true Value; and then, $\dot{y}$ being $= 2Ax\dot{x} + B\dot{x} + \frac{d\dot{x}}{a} \times \overline{M}|^{\frac{x}{a}}$ *, we ſhall, by ſubſtituting in the given Equation, have $cx^2\dot{x} + Ax^2\dot{x} + Bx\dot{x} + C\dot{x} + d\dot{x}M^{\frac{x}{a}} = 2Aax\dot{x} + Ba\dot{x} + d\dot{x}M^{\frac{x}{a}}$, and conſequently $\overline{c + A} \times x^2\dot{x} + \overline{B - 2Aa} \times x\dot{x} + \overline{C - Ba} \times \dot{x} = 0$. Whence $A = -c$ *, $B = -2ac$, $C = -2aac$; and conſequently $y = -c \times \overline{x^2 + 2ax + 2aa} + dM^{\frac{x}{a}}$. By the very ſame Way, the Value of y, in the Equation $cx^n\dot{x} + y\dot{x} = a\dot{y}$, will come out $= -c \times \overline{x^n + ax^{n-1} + n.\overline{n-1}.a^2x^{n-2} + n.\overline{n-1}.\overline{n-2}.a^3x^{n-3} + \&c.} + dM^{\frac{x}{a}}$.

* Art. 14.

* Art. 84.

266. But, what is a little remarkable, in theſe Equations, is, that the *Exponential* $dM^{\frac{x}{a}}$, tho' a variable Quantity, ſhould only ſerve, as it were, to correct the Fluent, or perform the Office of a conſtant Quantity. What I here mean will plainly appear, if it be conſidered, that the Equation $y = -c \times \overline{x^2 + 2ax + 2aa}$, where the ſaid *Exponential* is wanting, anſwers all the Conditions of the fluxional Equation firſt propoſed; which, upon Trial, will be found; and muſt needs be

the Caſe, ſeeing d may be, either, taken Nothing at all, or any Quantity at Pleaſure.

But the Equation $y = -c\times\overline{x^2+2ax+2a^2}$ (when $dM^{\frac{x}{a}}$ is wanting) cannot be corrected, in the uſual Way, ſo as to give $y=0$, when $x=0$; ſince, if any other conſtant Quantity, beſides $-2a^2c$ be introduced, the firſt Conditions will not be anſwer'd: The Correction muſt, therefore, be by the Exponential $dM^{\frac{x}{a}}$; and is thus.

Since $y = -cx^2 - 2cax - 2ca^2 + dM^{\frac{x}{a}}$, if y be taken $=0$ and $x=0$, then $-2ca^2 + dM^0 = 0$, or $d = 2ca^2$; and ſo the Equation, truly corrected, is $y = -c\times\overline{x^2+2ax+2a^2} + 2a^2cM^{\frac{x}{a}}$.

267. We come now to the laſt Method; namely, that of Infinite Serieſes; which, tho' leſs accurate, is vaſtly more comprehenſive, than *any* yet explained: The Manner of it is thus:

For the Quantity whoſe Value you would find, let an Infinite Series, conſiſting of the Powers of the other Quantity with unknown Coefficients, be aſſumed; which Series, together with its Fluxion, or Fluxions, muſt be ſubſtituted inſtead of their Equals in the given Equation; whence a new Equation will ariſe, from which, by comparing the homologous Terms, the aſſumed Coefficients, and conſequently the Value ſought, will be determined.

Thus, let the Equation $\frac{\dot{x}}{1+x} = \dot{y}$ (reducible to $\dot{x} - \dot{y} - x\dot{y} = 0$) be propoſed; to find x in Terms of y.
Then, aſſuming $x = Ay + By^2 + Cy^3 + Dy^4 + Ey^5$ &c.
We have $\dot{x} = A\dot{y} + 2By\dot{y} + 3Cy^2\dot{y} + 4Dy^3\dot{y} + 5Ey^4\dot{y} +$ &c.
Which Values being ſubſtituted in $\dot{x} - \dot{y} - x\dot{y} = 0$, we get

$$\left.\begin{array}{l} A\dot{y} + 2By\dot{y} + 3Cy^2\dot{y} + 4Dy^3\dot{y} + \text{\&c.} \\ -\dot{y} - Ay\dot{y} - By^2y \quad - Cy^3\dot{y} \quad - \text{\&c.} \end{array}\right\} = 0.$$

Therefore $A-1=0$, or $A=1$; $2B-A=0$, or $B=\frac{A}{2}=\frac{1}{2}$; $3C-B=0$, or $C=\frac{B}{3}=\frac{1}{2.3}$; $4D-C=0$, or $D=\frac{C}{4}=\frac{1}{2.3.4}$ &c.
And conſequently x $(Ay+By^2+Cy^3$ &c.$)$ $=y+\frac{y^2}{2}+\frac{y^3}{2.3}+\frac{y^4}{2.3.4}+\frac{y^5}{2.3.4.5}+$ &c.

Again, let it be required to find the Value of y, in the Equation $cx^2\dot{x}+y\dot{x}=a\dot{y}$, or $a\dot{y}-y\dot{x}-cx^2\dot{x}=0$. Here, aſſuming $y=Ax+Bx^2+Cx^3+Dx^4+Ex^5+Fx^6$ &c. and proceeding as before, we ſhall have

$$\left.\begin{array}{lllllll} aA\dot{x} & +2aBx\dot{x} & +3aCx^2\dot{x} & +4aDx^3\dot{x} & +5aEx^4\dot{x} & + & \&c. \\ 0 & -\ Ax\dot{x} & -\ Bx^2\dot{x} & -\ Cx^3\dot{x} & -\ Dx^4\dot{x} & - & \&c. \\ 0 & 0 & -\ cx^2\dot{x} & & & & \end{array}\right\} = 0$$

Whence $A=0$; $2aB=A=0$; $3aC=B+c=c$, or $C=\frac{c}{3a}$; $4aD=C=\frac{c}{3a}$, or $D=\frac{c}{3.4a^2}$; $5aE=D=\frac{c}{3.4a^2}$, or $E=\frac{c}{3.4.5a^3}$ &c. and conſequently y $(Ax+Bx^2+Cx^3+$ &c.$)$ $=\frac{cx^3}{3a}+\frac{cx^4}{3.4a^2}+\frac{cx^5}{3.4.5a^3}+\frac{cx^6}{3.4.5.6a^4}+$ &c.

268. It appears from this Example, that the Quantity to be found, will not always require all the Terms of the Series $Ax+Bx^2+Cx^3$ &c. And it may happen, in inumerable Caſes, that the Series to be aſſumed will demand a very different Law from *that* where the Exponents proceed according to the Terms of an arithmetical Progreſſion having Unity for the common Difference. And, indeed, the greateſt Difficulty we have here to encounter, is, to know what Kind of Series, with regard to its Exponents, ought to be aſſumed, ſo as to anſwer the Conditions of the Equation, without introducing more Terms than are actually neceſſary.

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The following Rules will be found very uſeful upon this Occaſion: Which, though they may become impracticable in certain particular Caſes, never take in any ſuperfluous Terms.

1°. *Having (if neceſſary) freed your Equation from Fractions and Surds, let the Quantity, whoſe Value is ſought, be ſuppoſed equal to ſome Power of the other Quantity with an unknown Exponent* (n); *and let that Power, together with its Fluxion, or Fluxions, be ſubſtituted for their (ſuppoſed) Equals in the given Equation.*

2°. *Let the leaſt Exponents of the variable, or indeterminate, Quantity, in the new Equation, thence ariſing, be put equal to each other: Whence the Value of the unknown Exponent* n *will be found.*

3°. *Subſtitute the Value of* n, *ſo found, in all the Exponents where* n *is concerned; and then take the Difference between one of the equal* ones, *above mention'd, and every other Exponent, of the variable Quantity, in the whole Equation.*

4°. *To theſe Differences, write down all the leaſt Numbers that can be compoſed out of them, by continual Addition, either to themſelves, or to one another; till you have, by that means, got, in the whole, as many different Terms, as you would have the required Series continued to.*

5°. *Laſtly, let each of thoſe Terms be increaſed by the Value of* n *(found by Rule 2.) and you will then have the Exponents of the Series to be aſſumed.*

EXAMPLE I.

269. *Let the Value of* x, *in the Equation* $a^2\dot{x}^2+x^2\dot{z}^2-a^2\dot{z}^2=0$, *be required.*

Firſt, by writing z^n for x, and $nz^{n-1}\dot{z}$ for $\dot{x}$, the Indices of z will be $2n-2$, $2n$, and 0 (which are determined by Inſpection, without regarding the Coefficients) whereof the two leaſt ($2n-2$ and 0) being put equal to each other, we here find $n=1$: Therefore, the Exponents being 0, 2, 0, the Differences (according to *Rule* 3.) are alſo 0, 2; from whence, by adding 2 continually, we get 0, 2, 4, 6, 8 *&c.* which (being each in-

increaſed by the Value of n) give 1, 3, 5, 7, 9 &c. for the Exponents in this Caſe.

Let, therefore, $x = Az + Bz^3 + Cz^5 + Dz^7 + \&c.$ Then, putting $\dot{z} = 1$, in order to facilitate the Operation, we ſhall have $\dot{x} = A + 3Bz^2 + 5Cz^4 + 7Dz^6 + \&c.$ which two Values being ſquared, and ſubſtituted in the given Equation, it will become

$$\left.\begin{array}{llllll} a^2A^2 + 6a^2ABz^2 & + 10a^2ACz^4 & + 14a^2ADz^6 & + \&c. \\ & + 9a^2B^2z^4 & + 30a^2BCz^6 & + \&c. \\ * \quad + A^2z^2 & + 2ABz^4 & + 2ACz^6 & + \&c. \\ -a^2 & & + B^2z^6 & + \&c. \end{array}\right\} = 0$$

Whence, $a^2A^2 = a^2$, and therefore $A = 1$; $6a^2B = -A$, and therefore $B = -\frac{1}{6a^2} = -\frac{1}{2.3a^2}$; $10a^2AC = -9a^2B^2 - 2AB = -B \times \overline{9a^2B + 2A} = -B \times \overline{-\frac{3}{2} + 2} = -\frac{B}{2} = \frac{1}{2.3.2a^2}$, and therefore $C = \frac{1}{2.3.4.5a^4}$; $14AD = -30a^2 \times -\frac{1}{6a^2} \times \frac{1}{120a^4} - 2 \times \frac{1}{120a^4} - \frac{1}{36a^4} = \frac{1}{24a^4} - \frac{1}{60a^4} - \frac{1}{36a^4} = -\frac{1}{360a^4}$, and therefore $D = -\frac{1}{14.360a^6} = -\frac{1}{2.3.4.5.6.7a^6}$; and, conſequently, $x = z - \frac{z^3}{2.3a^2} + \frac{z^5}{2.3.4.5a^4} - \frac{z^7}{2.3.4.5.6.7a^6}$ &c.

EXAMPLE II.

270. *Let the given Equation be* $a^2x\ddot{y} - 2a^2\dot{x}\dot{y} + ax\dot{x}^2 + x^3\ddot{y} = 0$; *to find* y.

Here, ſubſtituting x^n for y, the Exponents will be $n-1$, $n-1$, 1, and $n+1$; where, making $n-1=1$, we get $n=2$: Whence, the Differences being 0, 2, the Series to be aſſumed for y will be $Ax^2+Bx^4+Cx^6+Dx^8+Ex^{10}+$ *&c.* From which, making $\dot{x}=1$, we have $\dot{y}=2Ax+4Bx^3+6Cx^5+8Dx^7$ *&c.* and

$$\ddot{y}=2A+12Bx^2+30Cx^4+56Dx^6$$

And, theſe Values being ſubſtituted, the Equation becomes

$$\left.\begin{array}{l} 2A^2Ax+12a^2Bx^3+30a^2Cx^5+56a^2Dx^7+\&c. \\ -4a^2Ax-8a^2Bx^3-12a^2Cx^5-16a^2Dx^7+\&c. \\ +ax\quad+2Ax^3\quad+12Bx^5\quad+30Cx^7\quad+\&c. \end{array}\right\}=0$$

Therefore $A=-\frac{1}{2a}$; $B=-\frac{2A}{4a^2}=-\frac{1}{4a^3}$;

$C=-\frac{12B}{18a^2}=\frac{1}{6a^5}$; $D=-\frac{30C}{40a^2}=-\frac{1}{8a^7}$ *&c.*

and ſo $y=\frac{x^2}{2a}-\frac{x^4}{4a^3}+\frac{x^6}{6a^5}-\frac{x^8}{8a^7}+\frac{x^{10}}{10a^9}-$ *&c.*

Which Series is known to expreſs the Fluent of $\frac{ax\dot{x}}{a^2+x^2}$, or, $\frac{1}{2}a\times$ hyp. Log. $\frac{a^2+x^2}{aa}$: Conſequently y is alſo $=\frac{1}{2}a\times$ hyp. Log. $\frac{a^2+x^2}{a^2}$. In this manner, it comes to paſs, *that*, though we are obliged, in very complicated Caſes, to have recourſe to Infinite Serieſes, we are ſometimes able, at laſt, to give the Solution in finite Terms, or, at leaſt, by help of Logarithms, Sines and Tangents: Which will always happen when the Series can be ſummed, or is found to agree with that ariſing from ſome known Quantity.

271. Sometimes it happens, in Equations involving the higher Orders of Fluxions, that the Exponents, mention'd in *Rule* 2. whereof the leaſt ought to be made equal to each other, are ſo expreſſed, as to render ſuch an Equality impoſſible. When this is the Caſe, the Value of n, and the firſt Term of the required Series, can *only* be determined from the Nature of the Problem to which the Equation belongs. We know, in-

indeed, from the Equation itſelf, that n muſt be either equal to Nothing, or to ſome poſitive Integer, leſs than that expreſſing the Order of the higheſt Fluxion in the Equation: Becauſe the Term that has the leaſt Exponent, and which therefore cannot be compared with any other (being always affected by two or more of the Factors n, $n-1$, $n-2$, *&c.*) will then (one of thoſe Factors being $=0$) vaniſh intirely out of the Equation; which, thereby, is render'd poſſible.

When n and A are known, the reſt of the Terms will be found in the common Way, as in

EXAMPLE III.

Where the Equation propoſed is $y\dot{x}^2 + a\dot{x}\dot{y} - a^2\ddot{y} = 0$; *to find* y.

By ſuppoſing $\dot{x}=1$, and writing x^n for y, nx^{n-1} for $\dot{y}$, and $n \times \overline{n-1} \times x^{n-2}$ for $\ddot{y}$, we get $x^n + nax^{n-1} - n \times \overline{n-1} \times a^2x^{n-2}$: But it is plain that no two of the Indices of x can, *here*, be equal: The Value of n muſt therefore be either $=0$, or Unity (in both which Caſes the Term $-n \times \overline{n-1} \times a^2x^{n-2}$ vaniſhes) but I ſhall take the latter Value, and ſuppoſe the firſt Term of the Series to be Ax; then, the Differences of the foreſaid Exponents being 1 and 2, the Law of the Series will be expreſſed by 1, 2, 3, 4 *&c.* Whence, aſſuming $y = Ax + Bx^2 + Cx^3 + Dx^4$ *&c.* and proceeding as in the former Examples, y will be found $=$ A *into* $x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \frac{x^4}{8a^3} + \frac{x^5}{24a^4} + \frac{x^6}{90a^5}$ *&c.* or $=$A *into* $x + \frac{x^2}{2a} + \frac{2x^3}{2.3a^2} + \frac{3x^4}{2.3.4a^3} + \frac{5x^5}{2.3.4.5a^4} + \frac{8x^6}{2.3.4.5.6a^5} +$ *&c.* where the Law of Continuation is manifeſt, the Coefficient of every Numerator being compoſed by the Addition of the two preceding ones.

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272. It will be proper to obſerve here, that, in Equations like the two laſt propoſed, where the higher Orders of Fluxions are concerned, the Series expreſſing the Relation of the two Quantities muſt always be found in Terms of the Quantity flowing uniformly. And, that, if the Number of Dimenſions of the Fluxion of the ſaid Quantity, after Subſtitution, be not the ſame in every Term, the Equation itſelf, put down to be reſolved, is abſurd and impoſſible, and ſuch as never can ariſe in the Solution of any Problem. In all proper Equations the Number of fluxional Points (ſuppoſing the Powers of the Fluxions to be wrote without Indices) will be the ſame in every Term.

EXAMPLE IV.

273. *Where let the given Equation be* $a^3\dot{y}-ay^2\dot{x}+x^2y\dot{y}=x^3\dot{x}$; *to find* y.

By proceeding as uſual the Indices will here be $n-1$, $2n$, $2n+1$ and 3; whereof the leaſt (which can be no other than $n-1$ and 3) being compared, n will be given $=4$: And the Differences will therefore be 0, 5, 6; to which the Double of the Second and the Sum of the ſecond and third, &c. being put down, and then every Term increaſed by 4, there ariſes 4, 9, 10, 14, 15, 16, 19 &c. for the Exponents of the Series to be aſſumed for y.

Let therefore $y=Ax^4+Bx^9+Cx^{10}+Dx^{14}$ &c. then, making $\dot{x}=1$, $\dot{y}$ is $=4Ax^3+9Bx^8+10Cx^9+14Dx^{13}$ + &c.

And, by ſubſtituting theſe Values above, we have

$$\left.\begin{array}{l} 4a^3Ax^3+9a^3Bx^8+10a^3Cx^9+14a^3Dx^{13}+ \text{\&c.} \\ -x^3 \quad -aA^2x^8 \quad +4A^2x^9 \quad -2aABx^{13}+ \text{\&c.} \end{array}\right\}=0$$

Whence $A=\dfrac{1}{4a^3}$, $B=\dfrac{1}{144a^8}$, &c.

And * $y=\dfrac{x^4}{4a^3}-\dfrac{x^9}{144a^8}-\dfrac{x^{10}}{40a^9}+\dfrac{x^{14}}{4032a^{13}}$ &c.

---

* *If, for* y, *the Series* $Ax^4+Bx^5+Cx^6+Dx^7$ &c. *whoſe Exponents are in arithmetical Progreſſion, had been aſſumed, according to the Method of ſome very good Authors, no leſs than ſeven ſuperfluous Terms muſt have been introduced to obtain the four above given.*

274. Before I quit this Subject, it may not be amiſs to ſubjoin the following Remarks.

1°. If the indeterminate Quantities are great in reſpect to the given ones, a deſcending Series will, in moſt Caſes (where it is practicable) converge better than an aſcending one. To obtain ſuch a Series, compare the greateſt Exponents, mention'd in *Rule* 2. inſtead of the leaſt, and proceed according to the third and fourth Rules *, whence a Series of Numbers will be found; which, being ſucceſſively ſubtracted from the Value of n, you will have the Exponents of a deſcending Series.

Thus, let the common-algebraic Equation $a^3x+ax^3-a^3y-y^4=0$ be propounded; to find y, when x is great in compariſon of a.

Then, proceeding as uſual, the Exponents of the four Terms of the Equation will be 1, 3, n, $4n$; whereof the two greateſt ($4n$ and 3) being made equal, we get $n=\frac{3}{4}$; therefore the Differences are 0, 2 and $2\frac{1}{4}$; and the Numbers to be ſubſtracted from n, are 0, 2, $\frac{9}{4}$, 4, $\frac{17}{4}$, &c. Conſequently the Series to be aſſumed for y is $Ax^{\frac{3}{4}}+Bx^{\frac{-5}{4}}+Cx^{\frac{-6}{4}}+Dx^{\frac{-13}{4}}+$ &c. From whence y will be found $=a^{\frac{1}{4}}x^{\frac{3}{4}}+\dfrac{a^{\frac{9}{4}}}{4x^{\frac{5}{4}}}-\dfrac{a^{\frac{10}{4}}}{4x^{\frac{6}{4}}}-\dfrac{3a^{\frac{17}{4}}}{32x^{\frac{13}{4}}}$ &c.

2°. But, if the Quantity (x) in whoſe Terms the other is to be expreſſed, be neither much greater nor much ſmaller than the given Quantity (a), it will be proper to ſubſtitute for the Exceſs, or Defect, of the ſaid Quantity (x) above, or below, ſome given Quantity; ſo that, having, by this means, exterminated x, the Series ariſing from the new Equation (wherein the ſaid Exceſs, or Defect, is the converging Quantity) will have a due Rate of Convergency.

The Uſe of this is ſo obvious that it needs no Example, or farther Explanation.

3°. Laſtly, it will be proper to obſerve, that, if the Equation for the Value of A, ariſing from the firſt Column of homologous Terms, admits of two or more, equal

* Art. 268.

equal Roots (which is a Caſe that may, perhaps, never happen in Practice) all the foregoing Precepts will be inſufficient; unleſs the Equation alſo admits of ſome other Root, beſides the equal ones, whereby A may be more commodiouſly expreſſed. To determine the Exponents, in that particular Caſe, divide each of the Differences mention'd in *Rule* 3. by the Number of the equal Roots; and then proceed as uſual. The Reaſons of which, as well as of the Rules themſelves, I have long ago given elſewhere, and have not Room to repeat them here.

SCHOLIUM.

275. Although the Buſineſs of reverting Serieſes is not a Branch of the Doctrine of Fluxions, but, more properly, belongs to common *Algebra*; yet, as it is often uſeful where Fluxions are concerned, and falls under the general Rules illuſtrated in the foregoing Pages, I ſhall here add an Example or two on that Head.

Let, then, $ax+bx^2+cx^3+dx^4+ex^5$ &c. $=y$; to revert the Series, or, to find x in an Infinite Series expreſſed in the Powers of y.

Here, by writing y^n for x, the Indices of the Powers of y, in the Equation, will be n, $2n$, $3n$, &c. and 1; therefore $n=1$. and the Differences are 0, 1, 2, 3, 4, 5, &c. and ſo the Series to be aſſumed, in this Caſe, is $Ay+By^2+Cy^3+Dy^4$ &c. Which being involved and ſubſtituted for the reſpective Powers of x (neglecting, every where, all ſuch Powers of x and y as exceed the higheſt you would have the Series carry'd to) there ariſes

$$\left.\begin{array}{llllll}
aAy & +aBy^2 & +aCy^3 & +aDy^4 & & \&c. \\
* & +bA^2y^2 & +2bABy^3 & +2bACy^4 & \rbrace & \&c. \\
 & & & +bB^2y^4 & \rbrace & \&c. \\
* & * & +cA^3y^3 & +3cA^2By^4 & & \&c. \\
* & * & * & +dA^4y^4 & & \&c.
\end{array}\right\} = y$$

Whence, by comparing the homologous Terms, $A = \frac{1}{a}$; $B = -\frac{b}{a^3}$; $C \left(= -\frac{2bAB+cA^3}{a}\right) = \frac{2bb-ac}{a^5}$; $D \left(= -\frac{2bAC+bB^2+3cA^2B+dA^4}{a}\right) = \frac{5abc-5b^3-a^2d}{a^7}$, &c. and conſequently $x = \frac{y}{a} - \frac{by^2}{a^3} + \frac{2bb-ac}{a^5} \times y^3 - \frac{5b^3-5abc+a^2d}{a^7} \times y^4$ &c.

For an Inſtance of the Uſe of this Concluſion, let $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ &c. $= y$: Then, a being, in this Caſe, $=1$, $b = -\frac{1}{2}$, $c = \frac{1}{3}$, $d = -\frac{1}{4}$, &c. we ſhall, by ſubſtituting theſe Values, have $x = y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24}$ &c. From whence, when y is given, x will alſo be given; provided the Value of y be ſufficiently ſmall *. * Art. 92.

Example 2. Let there be given $ax+by+cx^2+dxy+ey^2+fx^3+gx^2y+hxy^2+iy^3+kx^4+lx^3y$ &c. $=0$; to find y.

By aſſuming $y=Ax+Bx^2+Cx^3+Dx^4$ &c. and proceeding as above, A will be found $= -\frac{a}{b}$, $B = -\frac{c+dA+eA^2}{b}$, $C = -\frac{dB+2eAB+f+gA+hA^2+iA^3}{b}$, $D = -\frac{dC+2eAC+eB^2+gB+2hAB+3iA^2B+k+lA+mA^2+nA^3+pA^4}{b}$, &c.

Example 3. Laſtly, let $x^m + bx^{m+p} + cx^{m+2p} + dx^{m+3p} +$ &c. $= z$.

Here, in order to determine the Form of the Series to be aſſumed, let z^n be wrote for x in the given Equation, according to the uſual Method; and then the Exponents, ſuppoſing z tranſpoſed, will be 1, nm, $nm + np$, $nm+2np$, $nm+3np$, &c. reſpectively; whereof the two leaſt (1 and nm) being made equal to each other, n is found $= \frac{1}{m}$; and the Differences are $\frac{p}{m}$, $\frac{2p}{m}$, $\frac{3p}{m}$, &c. Whence the Series to be aſſumed for x is $x^{\frac{1}{m}} + Bz^{\frac{1+p}{m}} + Cz^{\frac{1+2p}{m}} + Dz^{\frac{1+3p}{m}} +$ &c. (for it is evident, by Inſpection, that the Coefficient (A) of the firſt Term muſt here be an Unit.) This Series being therefore raiſed to the ſeveral Powers of x, in the given Equation, *by Art.* 108. and the Coefficients of the homologous Terms in the new Equation compared together, it will be found that, $B = -\frac{b}{m}$, $C = \frac{\overline{1+m+2p}\times bb-2mc}{2m^2}$, $D = -\frac{\overline{2m^2+9mp+9p^2+3m+6p+1}\times b^3}{6m^3} + \frac{\overline{1+m+3p}\times bc}{m^2} - \frac{d}{m}$, &c.

From the general Value of x, found above, innumerable Theorems, for reverting particular Forms of Serieſes, may be deduced.

Thus, if $x + bx^2 + cx^3 + dx^4$, &c. $= z$; then (m being $= 1$ and $n = 1$) x is $= z - bz^2 + \overline{2bb-c}\times z^3 - \overline{5b^3-5bc+d}\times z^4$ &c.

And, if $x+bx^3+cx^5+dx^7+$ &c. $=z$; (m being $=1$, and $p=2$) $x=z-bz^3+\overline{3bb-c}\times z^5-\overline{12b^3-8cb+d}\times z^7$ &c.

Alſo, if $x^{\frac{1}{2}}+bx^{\frac{3}{2}}+cx^{\frac{5}{2}}+dx^{\frac{7}{2}}$ &c. $=z$; then (m being $=\frac{1}{2}$ and $p=1$) $x=z^2-2bz^4+\overline{7bb-2c}\times z^6-\overline{30b^3-18bc+2d}\times z^8$ &c. &c.

276. It may be obſerved that, in all theſe Forms of Serieſes, the firſt Term is without a Coefficient (which renders the Concluſion much more ſimple.) Therefore, when the Series to be reverted has a Co-efficient in its firſt Term, the whole Equation muſt be firſt of all divided thereby: Thus, if the Equation was $3x-6x^2+8x^3-13x^4$ &c. $=y$; by dividing the whole by 3 it will become $x-2x^2+\frac{8x^3}{3}-\frac{13x^4}{3}$ &c. $=\frac{1}{3}y$: Where, putting $z=\frac{1}{3}y$, we have, *by Form.* 1. $x=z+2z^2+\frac{16}{3}z^3$ &c. $=\frac{y}{3}+\frac{2y^2}{9}+\frac{16y^3}{81}$ &c.

SECTION III.

Of the Compariſon of Fluents, or the Manner of finding one Fluent from another.

277. WE have, already, pointed out the moſt remarkable Forms of Fluxions whoſe Fluents are explicable in finite Terms *; and alſo ſhewn the Uſe of Infinite Serieſes in approximating the Values of ſuch Fluents as do not come under any of thoſe Forms ‡: But this laſt Method (as is before hinted) being troubleſome, and attended with many Obſtacles; Mathematicians have therefore invented, and ſhewn, the Way of deriving one Fluent from another: Which is of good Advantage when the Fluent

* Art. 77. 78. 83. 84. and 85.

‡ Art. 99.

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ſought can be referred to one, like thoſe in *Art.* 126. and 142. expreſſing the Logarithm of a Number, or the Arch of a Circle; ſince the Trouble of an infinite Series is, then, avoided.

As the Subject here propoſed is of ſuch a Nature, that it would be very tedious and difficult, if not altogether impracticable, to lay down Rules and Precepts for all the various Caſes; I ſhall deliver, what I have to offer thereon, by way of *Problems*; beginning with ſome very eaſy ones, for the Sake of the *young Proficient.*

PROB. I.

278. *The Fluent of* $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ *being given (by Art.* 126.) *'tis propoſed to find, from thence, the Fluent of* $\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$.

Let both the Numerator and Denominator of $\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$, be multiply'd by x, ſo that the Quantity without the Vinculum, in the Fluxion, $\frac{x^3\dot{x}}{\sqrt{a^2x^2+x^4}}$, thus transformed, may become ſome conſtant Part of the Fluxion of the higheſt Term under the Vinculum: Which Part, in this Caſe, being $\frac{1}{4}$, let $\frac{1}{4}$ of the Fluxion of the firſt Term under the Vinculum (or $\frac{1}{2}a^2x\dot{x}$) be therefore added to the Numerator, in order to have the *Whole*, $\frac{\frac{1}{2}a^2x\dot{x}+x^3\dot{x}}{\sqrt{a^2x^2+x^4}}$, a *complete* Fluxion; and then the Fluent thereof, by the common Rule *, will be $\frac{1}{2}\sqrt{a^2x^2+x^4} = \frac{1}{2}x\sqrt{a^2+x^2}$: But, from *this*, we are now to deduct the Fluent of the Quantity $\frac{\frac{1}{2}a^2x\dot{x}}{\sqrt{a^2x^2+x^4}}$ $\left(= \frac{\frac{1}{2}a^2\dot{x}}{\sqrt{a^2+x^2}}\right)$ that was added: Which Fluent, as

that

* Art. 77.

that of $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ is given $=$ *hyp. Log.* $\overline{x+\sqrt{a^2+x^2}}$ *, will be $= \frac{1}{2} a^2 \times$ *hyp. Log.* $\overline{x+\sqrt{a^2+x^2}}$; and conſequently the Fluent ſought $= \frac{1}{2} x\sqrt{a^2+x^2} - \frac{1}{2} a^2 \times$ *hyp. Log.* $\overline{x+\sqrt{a^2+x^2}}$. *Q. E. I.*

PROB. II.

279. *Let it be propoſed to find the Fluent of* $\frac{x^2\dot{x}}{\sqrt{a^2-x^2}}$, *from that of* $\frac{\dot{x}}{\sqrt{a^2-x^2}}$; *given by* Art. 142.

By proceeding as above, and adding $-\frac{1}{2} a^2x\dot{x}$ to the Numerator, we have $-\frac{\frac{1}{2} a^2x\dot{x} - x^3\dot{x}}{\sqrt{a^2x^2-x^4}}$; whereof the Fluent, by the *common Rule*, is $-\frac{1}{2}\sqrt{a^2x^2-x^4}$ $(= -\frac{1}{2} x\sqrt{a^2-x^2})$: From which deducting the Fluent of $-\frac{\frac{1}{2} a^2x\dot{x}}{\sqrt{a^2x^2-x^4}}$, or $-\frac{\frac{1}{2} a^2\dot{x}}{\sqrt{a^2-x^2}}$ (given $= -\frac{1}{2} a^2 \times$ Arc (A) whoſe Radius is Unity and Sine $= \frac{x}{a}$ *) there comes out $\frac{1}{2} a^2 A - \frac{1}{2} x\sqrt{a^2-x^2}$.

Q. E. I.

280. In the ſame Manner, if the Power without the Vinculum, in the Expreſſion whoſe Fluent is ſought, exceeds *that* in the other Expreſſion given, by the Exponent under the Vinculum, or by any *Multiple* of it, the required Fluent may be determined, by one, or by ſeveral Operations, according to the Value of the ſaid *Multiple*.

Thus, if the Fluent of $\frac{x^4\dot{x}}{\sqrt{a^2-x^2}}$ was ſought; then, becauſe the Index of x, without the *Vinculum*, exceeds

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* Art. 126.

* Art. 142.

that in $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$ by twice the Exponent under the *Vinculum*, the required Fluent may be had from that of $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$, at two Operations; by the firſt whereof, we have already found the Fluent of $\frac{x^2\dot{x}}{\sqrt{a^2 - x^2}}$ to be $= \frac{1}{2} a^2 A - \frac{1}{2} x \sqrt{a^2 - x^2}$: Whence, putting this Value $= B$, and proceeding as before, we alſo get $-\frac{1}{4}\sqrt{a^2x^6 - x^8} + \frac{3}{4} a^2 B = -\frac{1}{4} x^3 \sqrt{a^2 - x^2} - \frac{3a^2x}{8} \sqrt{a^2 - x^2} + \frac{3a^4 A}{8} = \frac{3a^4 A - \overline{2xx + 3aa} \times x\sqrt{a^2 - x^2}}{8} =$ the true Fluent of $\frac{x^4\dot{x}}{\sqrt{a^2 - x^2}}$.

PROB. III.

281. *Suppoſing the Fluent of* $\overline{a + cz^n}^{m} \times z^{pn-1}\dot{z}$ *to be given* $= A$, *to find the Fluent of* $\overline{a + cz^n}^{m} \times z^{pn+n-1}\dot{z} = \dot{B}$ (*where the Exponent of* z, *without the* Vinculum *is increaſed by the Exponent under the* Vinculum.)

Let the Part affected by the *Vinculum* be multiply'd by z^{mq}, and the Part without be divided by the ſame Quantity; then our Fluxion will be transformed to $\overline{az^q + cz^{n+q}}^{m} \times z^{pn+n-mq-1}\dot{z} = \dot{B}$: Where let q be, now, ſo taken that the Exponent $(n+q)$ of the higheſt Power of z under the *Vinculum* may be equal to $(pn+n-mq)$ that of the Power without the *Vinculum* $+ 1$; that is, let $q = \frac{pn}{m+1}$: Then (*by Art.* 77. if the firſt Term under the *Vinculum* was conſtant, the Fluent of the ſaid Expreſſion, or its Equal $\overline{az^q+cz^{n+q}}^{\,m}\times z^{n+q-1}\dot{z}$, would be had $=\dfrac{\overline{az^q+cz^{n+q}}^{\,m+1}}{\overline{m+1}\times\overline{n+q}\times c}$. But the Fluxion hereof, ſuppoſing both Terms to be variable (as they actually are) is $\overline{az^q+cz^{n+q}}^{\,m}\times z^{n+q-1}\dot{z}+\dfrac{qa}{\overline{n+q}\times c}\times\overline{az^q+cz^{n+q}}^{\,m}\times z^{q-1}\dot{z}$ (by the common Rule.) Therefore $\dfrac{\overline{az^q+cz^{n+q}}^{\,m+1}}{\overline{m+1}\times\overline{n+q}\times c}-\dfrac{qa}{\overline{n+q}\times c}\times$ *Flu. of* $\overline{az^q+cz^{n+q}}^{\,m}\times z^{q-1}\dot{z}=B$; that is, $\dfrac{\overline{a+cz^n}^{\,m+1}\times z^{qm+q}}{\overline{m+1}\times\overline{n+q}\times c}-\dfrac{qa}{\overline{n+q}\times c}\times$ *Flu.* $\overline{a+cz^n}^{\,m}\times z^{mq+q-1}\dot{z}=B$; or, by ſubſtituting for q, $\dfrac{\overline{a+cz^n}^{\,m+1}\times z^{pn}}{\overline{m+p+1}\times nc}-\dfrac{pa}{\overline{m+p+1}\times c}\times$ *Flu.* $\overline{a+cz^n}^{\,m}\times z^{pn-1}\dot{z}=B$: But the *Flu. of* $\overline{a+cz^n}^{\,m}\times z^{pn-1}\dot{z}$ is given $=A$; therefore, laſtly, $\dfrac{\overline{a+cz^n}^{\,m+1}\times z^{pn}}{\overline{m+p+1}\times nc}-\dfrac{paA}{\overline{m+p+1}\times c}=B$. *Q. E. I.*

282. If the Quantity under the *Vinculum* be a Multinomial, $a+cz^n+dz^{2n}+ez^{3n}$ *&c.* Then, ſince the Fluxion of $\overline{a+cz^n+dz^{2n}+ez^{3n}\ \&c.}^{\,m+1}\times z^{pn}$ is $\overline{m+1}\times\overline{ncz^{n-1}\dot{z}+2ndz^{2n-1}\dot{z}+3nez^{3n-1}\dot{z}\ \&c.}\times$

$$\overline{a+cz^{n}+dz^{2n}\ \&c.}^{\,m}\times z^{pn}+\overline{a+cz^{n}+dz^{2n}\ \&c.}^{\,m+1}$$
$$\times pnz^{pn-1}\dot{z}=$$
$$\left\{\begin{array}{l} *\qquad\quad \overline{m+1}\times ncz^{pn+n-1}\dot{z}+\overline{m+1}\times 2ndz^{pn+2n-1}\dot{z}\ \&c. \\ pnaz^{pn-1}\dot{z}+pncz^{pn+n-1}\dot{z}+pndz^{pn+2n-1}\dot{z}\ \&c. \end{array}\right\}$$

$\times\overline{a+cz^{n}+dz^{2n}\ \&c.}^{\,m}$, it is evident, that, if the Fluents of $z^{pn-1}\dot{z}$, $z^{pn+n-1}\dot{z}$, $z^{pn+2n-1}\dot{z}$ &c. drawn into the general Multiplicator $\overline{a+cz^{n}+dz^{2n}\ \&c.}^{\,m}$, be denoted by *A*, *B*, *C*, *D*, &c. the Fluent of the Whole-Quantity exhibited above (which Fluent is $\overline{a+cz^{n}+dz^{2n}+ez^{3n}\ \&c.}^{\,m+1}\times z^{pn}$) will alſo be expreſſed by $pnaA+\overline{p+m+1}\times ncB+\overline{p+2m+2}\times ndC+\overline{p+3m+3}\times neD$ &c. Therefore, if there be given as many of the Fluents *A*, *B*, *C*, *D* &c. as there are Terms in $a+cz^{n}+dz^{2n}+ez^{3n}$ &c. *minus* one, that other Fluent, be it which it will, will alſo be given from hence. Thus, if $d=0$, $e=0$, &c. and the Value of *A* be given, we ſhall have $\overline{a+cz^{n}}^{\,m+1}\times z^{pn}=pnaA+\overline{p+m+1}\times ncB$; and conſequently $B=\dfrac{\overline{a+cz^{n}}^{\,m+1}\times z^{pn}}{\overline{p+m+1}\times nc}-\dfrac{paA}{\overline{p+m+1}\times c}$, the very ſame as before.

PROB. IV.

283. *The Fluent of* $\overline{a+cz^{n}}^{\,m}\times z^{pn-1}\dot{z}$ *being given (as in the preceding Problem) to determine, from thence, the Fluent of* $\overline{a+cz^{n}}^{\,m}\times z^{pn+vn-1}\dot{z}$; *ſuppoſing* v *to denote a whole poſitive Number.*

Let $\overline{a+cz^n}|^{m+1}$ be denoted by M; alſo put $p+1=\overset{\prime}{p}$, $\overset{\prime}{p}+1$ $(p+2)=\overset{\prime\prime}{p}$, $\overset{\prime\prime}{p}+1$ $(p+3)=\overset{\prime\prime\prime}{p}$ &c. and let the Fluents of $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$, $\overline{a+cz^n}|^m \times z^{\overset{\prime}{p}n-1}\dot{z}$, $\overline{a+cz^n}|^m \times z^{\overset{\prime\prime}{p}n-1}\dot{z}$, $\overline{a+cz^n}|^m \times z^{\overset{\prime\prime\prime}{p}n-1}\dot{z}$, &c. be repreſented by A, B, C, D, &c. reſpectively. Then, ſince $\frac{Mz^{pn}}{\overline{m+p+1}\times nc} - \frac{paA}{\overline{m+p+1}\times c} = B$ (*by the preceding Prob.*) it follows, from the very ſame Argument, that

$$\frac{Mz^{\overset{\prime}{p}n}}{\overline{m+\overset{\prime}{p}+1}\times nc} - \frac{\overset{\prime}{p}aB}{\overline{m+\overset{\prime}{p}+1}\times c} = C$$

$$\frac{Mz^{\overset{\prime\prime}{p}n}}{\overline{m+\overset{\prime\prime}{p}+1}\times nc} - \frac{\overset{\prime\prime}{p}aC}{\overline{m+\overset{\prime\prime}{p}+1}\times c} = D$$

&c. &c.

Hence, by writing the Value of B in the ſecond Equation, we have $\frac{Mz^{\overset{\prime}{p}n}}{\overline{m+\overset{\prime}{p}+1}\times nc} - \frac{\overset{\prime}{p}aMz^{pn}}{\overline{m+\overset{\prime}{p}+1}\times\overline{m+p+1}\times ncc} + \frac{\overset{\prime}{p}pa^2A}{\overline{m+p+1}\times\overline{m+\overset{\prime}{p}+1}\times c^2} = C$. In the ſame Manner, by ſubſtituting this Value for C in the 3d Equation, we get

$$\frac{Mz^{\overset{\prime\prime}{p}n}}{\overline{m+\overset{\prime\prime}{p}+1}\times nc} - \frac{\overset{\prime\prime}{p}aMz^{\overset{\prime}{p}n}}{\overline{m+\overset{\prime\prime}{p}+1}\times\overline{m+\overset{\prime}{p}+1}\times nc^2} +$$

$$\frac{\overset{\prime\prime}{p}\overset{\prime}{p}pa^2Mz^{pn}}{\overline{m+\overset{\prime\prime}{p}+1}\times\overline{m+\overset{\prime}{p}+1}\times\overline{m+p+1}\times nc^3}-\frac{p\overset{\prime}{p}\overset{\prime\prime}{p}a^3A}{\overline{m+p+1}\times\overline{m+\overset{\prime}{p}+1}\times\overline{m+\overset{\prime\prime}{p}+1}\times c^3}=D.$$

Where the Law of Continuation is manifeſt; and from whence it appears that the Value of any of the Quantities B, C, D, E. *&c.* or the Fluent expreſſed in a general Manner, will be $\frac{Mz^{qn}}{\overline{m+q+1}\times nc}-\frac{qaMz^{\overline{q-1}\times n}}{\overline{m+q+1}\times\overline{m+q}\times ncc}+\frac{q\times\overline{q-1}\times a^2Mz^{\overline{q-2}\times n}}{\overline{m+q+1}\times\overline{m+q}\times\overline{m+q-1}\times nc^3}\,(v)\pm\frac{p\times\overline{p+1}\times\overline{p+2}\times\overline{p+3}\,(v)\times a^vA}{\overline{m+p+1}\times\overline{m+p+2}\times\overline{m+p+3}\,(v)\times c^v}$; or, $\frac{\overline{a+cz^n}^{m+1}\times z^{pn}}{\overline{s+1}\times nc}\times\overline{z^{vn-n}-\frac{qaz^{vn-2n}}{sc}+\frac{q.\overline{q-1}\times a^2z^{vn-3n}}{s.\overline{s-1}\times c^2}-\frac{q.\overline{q-1}.\overline{q-2}\times a^3z^{vn-4n}}{s.\overline{s-1}.\overline{s-2}\times c^3}\,(v)}\pm\frac{p}{t}\times\frac{p+1}{t+1}\times\frac{p+2}{t+2}\times\frac{p+3}{t+3}\,(v)\times\frac{a^vA}{c^v}$: Where, $A=$ Fluent of $\overline{a+cz^n}^m\times z^{pn-1}\dot{z}$, $q=p+v-1$, $s=q+m$, $t=p+m+1$; and where the Sign of the laſt Term (in which A is found) muſt be taken $+$ or $-$ according as v is an even or odd Number: Note, alſo, that the Parentheſis (v) is put to expreſs the Number of Terms, or Factors, to which the Series, or Product, preceding it, is to be continued. The like Notation is to be underſtood in other Caſes of the ſame Kind, when they hereafter occur.

The

The ſame otherwiſe.

284. Let $q = p + v - 1$, and let $\overline{a+cz^n}|^{m+1} \times \overline{Rz^{qn}+Sz^{qn-n}+Tz^{qn-2n}\ldots\ldots+\Delta z^{pn}} + \beta A$, be aſſumed for the Fluent ſought: Then, by taking the Fluxion thereof, you will have $\overline{m+1}\times ncz^{n-1}\dot{z} \times \overline{a+cz^n}|^{m} \times \overline{Rz^{qn}+Sz^{qn-n}\ldots\ldots+\Delta z^{pn}} + \overline{a+cz^n}|^{m+1} \times \overline{qn\dot{z}Rz^{qn-1}+\overline{qn-n}\times\dot{z}Sz^{qn-n-1}\ldots\ldots+pn\dot{z}\Delta z^{pn-1}}$ $+ \beta \times \overline{a+cz^n}|^{m} \times z^{pn-1}\dot{z}$; which muſt be $= \overline{a+cz^n}|^{m} \times z^{pn+vn-1}\dot{z}$ (or $\overline{a+cz^n}|^{m} \times z^{qn+n-1}\dot{z}$) the Fluxion propoſed: Whence, dividing the whole Equation by $\overline{a+cz^n}|^{m} \times z^{n-1}\dot{z}$, and tranſpoſing, there comes out

$$\left.\begin{array}{l} \overline{m+1}\times nc\times \overline{Rz^{qn}+Sz^{qn-n}+Tz^{qn-2n}\ldots\ldots+\Delta z^{pn}} \\ \overline{a+cz^n}\times \overline{qnRz^{qn-n}+\overline{qn-n}\times Sz^{qn-2n}\ldots\ldots+pn\Delta z^{pn-n}} \\ \qquad -z^{qn} \qquad\quad * \qquad\quad * \qquad\quad +\beta z^{pn-n} \end{array}\right\} \begin{array}{c} \| \\ 0 \end{array}$$

Which, reduced, and the homologous Terms united, becomes

$$\left.\begin{array}{r} \overline{m+q+1}\times ncR \\ -1 \end{array}\right\} \times z^{qn} + \left.\begin{array}{r} \overline{m+q}\times ncS \\ +qnaR \end{array}\right\} \times z^{qn-n} +$$

$$\left.\begin{array}{r} \overline{m+q-1}\times ncT \\ +\overline{qn-n}\times aS \end{array}\right\} \times z^{qn-2n} \ldots\ldots\ldots \left.\begin{array}{l} +pna\Delta \\ +\beta \end{array}\right\} \times z^{pn-n}$$

$= 0$: Where, by making $\overline{m+q+1}\times ncR - 1 = 0$, $\overline{m+q}\times ncS + qnaR = 0$, &c. we have $R = \frac{1}{\overline{m+q+1}\times cn}$, $S = -\frac{qaR}{\overline{m+q}\times c}$, $T = -\frac{\overline{q-1}\times aS}{\overline{m+q-1}\times c}$; or (putting

$m+q=s$) $R=\frac{1}{\overline{s+1}\times nc}$, $S=-\frac{qaR}{sc}=-\frac{qa}{\overline{s+1}\times snc^2}$,

$T=-\frac{\overline{q-1}\times aS}{\overline{s-1}\times c}=\frac{q\times\overline{q-1}\times a^2}{\overline{s+1}\times s\times\overline{s-1}\times nc^3}$, &c.

Where, becauſe the Exponent of the firſt Term of the Equation is qn ($pn+vn-n$) and that of the laſt Term (in which Δ and β are concerned) $=pn$, it follows that the Number of Coefficients to be taken as above (whereof Δ is the laſt) is expreſſed by v: From which laſt, the Value of β is given $=-pna\Delta$.

But, from the Law of the ſaid Coefficients, R, S, ...Δ, it appears that the Value of Δ (whoſe Place from the Beginning is denoted by v) will be $=\pm$ $\frac{q.\overline{q-1}.\overline{q-2}\ldots\ldots\overline{q-v+2}}{\overline{s+1}.s.\overline{s-1}\ldots\ldots\overline{s-v+2}}\times\frac{a^{n-1}}{nc^n}=\pm$ $\frac{q.\overline{q-1}.\overline{q-2}\ldots\ldots\overline{p+1}}{\overline{s+1}.s.\overline{s-1}\ldots\ldots\overline{p+m+1}}\times\frac{a^{n-1}}{nc^n}$: And therefore β $(=-pna\Delta)=\pm\frac{q.\overline{q-1}.\overline{q-2}\ldots\ldots\overline{p+1}.p}{\overline{s+1}.s.\overline{s-1}\ldots\ldots\overline{p+m+1}}$ $\times\frac{a^n}{c^n}=\pm\frac{p.\overline{p+1}.\overline{p+2}.\overline{p+3}\,(v)}{t.\overline{t+1}.\overline{t+2}.\overline{t+3}\,(v)}\times\frac{a^n}{c^n}$ (putting $p+m+1=t$, as before.) Now, if the ſeveral Values of R, S. T...... and β, thus found, be ſubſtituted in the aſſumed Expreſſion, you will have the very ſame Concluſion as in the preceding Article.

COROLLARY I.

285. Since q is $=p+v-1$, the Fluent $\overline{a+cz^n}\big|^{m+1}\times$ $\overline{Rz^{qn}+Sz^{qn-n}\ldots+\Delta z^{pn}}+\beta A$, given above, may be expreſſed by $N\times\overline{Rz^{vn-n}+Sz^{vn-2n}+Tz^{vn-3n}}$ (v) $+\beta A$; where $N=\overline{a+cz^n}\big|^{m+1}\times z^{pn}$, $R=$

$\frac{1}{\overline{m+p+v}\times nc}$, $S = -\frac{\overline{p+v-1}.aR}{\overline{m+p+v-1}.c}$, $T = -\frac{\overline{p+v-2}.aS}{\overline{m+p+v-2}.c}$: And, where the Coefficient (β) of the given Fluent (A) will always be expreſſed by the laſt of the Quantities R, S, T . . . Δ, multiply'd by $-pna$: This is evident, becauſe it is found that $\beta = -pna\Delta$. And the ſame thing will alſo appear from the ſeveral particular Caſes (*in Art.* 283.) for the Values of B, C and D: In each of which the Coefficient of the laſt Term (where A is concern'd) is to *that* of the Term immediately preceding it, in the conſtant Ratio of pa to $\frac{1}{n}$, or of pna to Unity.

COROLLARY II.

286. If the Value of c be negative, the general Fluent (*in Art.* 283.) when $a+cz^n = 0$ (provided $m+1$, n, and p be poſitive) will become barely $= \pm\frac{p}{t}\times\frac{p+1}{t+1}\times\frac{p+2}{t+2}\,(v)\times\frac{a^vA}{c^v}$; becauſe, in this Circumſtance, all the Terms multiply'd by $\overline{a+cz^n}^{m+1}$ intirely vaniſh. If, therefore, b be wrote for $-c$ (to render the Expreſſion more commodious) we ſhall have $\frac{p}{t}\times\frac{p+1}{t+1}\times\frac{p+2}{t+2}\,(v)\times\frac{a^vA}{b^v}$ for the true Fluent of $\overline{a-bz^n}^m\times z^{pn+vn-1}\dot{z}$, generated while bz^n, from Nothing, becomes $=a$: Where A denotes the Fluent of $\overline{a-bz^n}^m \times z^{pn-1}\dot{z}$, generated in the ſame time; and where $t=$

$t = p + m + 1$. Hence it follows that the Fluent of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z} \times \overline{e+fz^n+gz^{2n}+hz^{3n}\ \&c.}$ (where e, f, g are any given Quantities) will be $= A \times \overline{e + \frac{paf}{tb} + \frac{p\cdot\overline{p+1}\cdot a^2g}{t\cdot\overline{t+1}\cdot b^2} + \frac{p\,\overline{p+1}\cdot\overline{p+2}\cdot a^3h}{t\cdot\overline{t+1}\cdot\overline{t+2}\cdot b^3} + \&c.}$ in the foremention'd Circumſtance.

PROB. V.

287. *The Fluent* (A) of $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$ *being given, to find the Fluent of* $\overline{a+cz^n}|^{m+r} \times z^{pn-1}\dot{z}$; *ſuppoſing* r *to denote a whole poſitive Number.*

Since $\overline{a+cz^n}|^{m+1} = \overline{a+cz^n}|^m \times \overline{a+cz^n}$, it is evident that $\overline{a+cz^n}|^{m+1} \times z^{pn-1}\dot{z} = \overline{a+cz^n}|^m \times az^{pn-1}\dot{z} + \overline{a+cz^n}|^m \times cz^{pn+n-1}\dot{z}$: Whoſe Fluent (*by Prob.* 3.) is $aA + \frac{\overline{a+cz^n}|^{m+1} \times z^{pn}}{\overline{m+p+1} \times n} - \frac{paA}{m+p+1} = \frac{\overline{a+cz^n}|^{m+1} \times z^{pn}}{\overline{p+m+1} \times n} + \frac{\overline{m+1} \times aA}{p+m+1}$. In like Manner, if this Fluent, of $\overline{a+cz^n}|^{m+1} \times z^{pn-1}\dot{z}$, be denoted by B, that of $\overline{a+cz^n}|^{m+2} \times z^{pn-1}\dot{z}$ by C, &c. it will appear that $\frac{\overline{a+cz^n}|^{m+2} \times z^{pn}}{\overline{p+m+2} \times n} + \frac{\overline{m+2} \times aB}{p+m+2} = C$; $\frac{\overline{a+cz^n}|^{m+3} \times z^{pn}}{\overline{p+m+3} \times n} + \frac{\overline{m+3} \times aC}{p+m+3} = D$, &c. Whence, by ſubſtituting theſe Values, one by one, as in the preceding Problem, and putting $Q = a + cz^n$, we get

$$C = \frac{Q^{m+2}\, z^{pn}}{\overline{p+m+2}.n} + \frac{\overline{m+2}\times a Q^{m+1}\, z^{pn}}{\overline{p+m+2}.\overline{p+m+1}.n} + \frac{\overline{m+2}.\overline{m+1}\times a^2 A}{\overline{p+m+2}.\overline{p+m+1}}; \quad D = \frac{Q^{m+3}\, z^{pn}}{\overline{p+m+3}.n} + \frac{\overline{m+3}\times a Q^{m+2}\, z^{pn}}{\overline{p+m+3}\,\overline{p+m+2}.n} + \frac{\overline{m+3}.\overline{m+2}\times a^2 Q^{m+1}\, z^{pn}}{\overline{p+m+3}.\overline{p+m+2}.\overline{p+m+1}.n} + \frac{\overline{m+3}.\overline{m+2}.\overline{m+1}.a^3 A}{\overline{p+m+3}.\overline{p+m+2}.\overline{p+m+1}},$$

&c. Whence it is evident, by Inſpection, that the Fluent of $\overline{a+cz^n}^{m+r} \times z^{pn-1}\dot{z}$, expreſſed in a general Manner, will be

$$\frac{Q^{m+r}\, z^{pn}}{\overline{p+m+r}.n} + \frac{\overline{m+r}\times a Q^{m+r-1}\, z^{pn}}{\overline{p+m+r}\times\overline{p+m+r-1}.n}$$

&c. Which, by putting $m+r=f$, $p+m+r=g$, and making $Q^{m+1}\times z^{pn}$ a general Multiplicator, will be reduced to

$$Q^{m+1}\times z^{pn}\times\overline{\frac{Q^{r-1}}{gn}+\frac{f\times a Q^{r-2}}{g.\overline{g-1}.n}+\frac{f.\overline{f-1}\times a^2 Q^{r-3}}{g.\overline{g-1}.\overline{g-2}.n}\ (r)} + \frac{m+1}{p+m+1}\times\frac{m+2}{p+m+2}\times\frac{m+3}{p+m+3}\ (r)\ a^r A;$$

where it appears (from the foregoing Values of B, C, and D) that the Coefficient of A is always equal to the laſt Term of the preceding Series, multiply'd by $\overline{m+1}\times na$ (inſtead of $Q^{m+1}\, z^{pn}$.)

Q. E. I.

Corollary.

288. If c be negative, ſo that Q, or its Equal, $a+cz^n$, may become equal to Nothing, the Fluent will,

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in that Circumſtance, be barely $= \frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2} \times \frac{m+3}{p+m+3} (r) \times a^r A$; provided the Values of $m+1$; p, and n are poſitive: Or, if c, p and n be poſitive, and $m+r+p$ negative, the ſame Expreſſion will exhibit the true Value of the whole Fluent, generated while z, from Nothing, becomes infinite.

PROB. VI.

289. *The ſame being given as in the preceding Problems; 'tis propoſed to find the Fluent of* $\overline{a+cz^n}\,^{m-r} \times z^{pn-1}\dot{z}$.

If $-r$ be wrote inſtead of r, in the laſt Article, we ſhall have $m-r=f$, $p+m-r=g$, and $\mathcal{Q}^{m+1} z^{pn} \times \frac{\mathcal{Q}^{-r-1}}{gn} + \frac{f \times .a\mathcal{Q}^{-r-2}}{g.\overline{g-1}.n} (-r) + \frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2} (-r) \times a^{-r} A$, expreſſing the required Fluent in this Caſe.

But $\frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2}$ &c. continued to $-r$ Factors, ſignifies the ſame thing as the Product continued downwards, or the contrary way, to r Factors, according to the ſame Law: And therefore is $= \frac{p+m}{m} \times \frac{p+m-1}{m-1} \times \frac{p+m-2}{m-2} (r)$. After the ſame Manner we have $\frac{\mathcal{Q}^{-r-1}}{gn} + \frac{f \times a\mathcal{Q}^{-r-2}}{g.\overline{g-1}.n} (-r) =$

$$- \frac{\mathcal{Q}^{-r}}{\overline{f+1}.na} - \frac{\overline{g+1}.\mathcal{Q}^{-r+1}}{\overline{f+1}.\overline{f+2}.na^2} - \frac{\overline{g+1}.\overline{g+2}.\mathcal{Q}^{-r+2}}{\overline{f+1}.\overline{f+2}.\overline{f+3}.na^3} (r)$$

(r) and conſequently the Fluent itſelf $= \mathcal{Q}^{m+1} z^{pn} \times$

$$\overline{\frac{-\mathcal{Q}^{-r}}{\overline{f+1}.na} - \frac{\overline{g+1}.\mathcal{Q}^{1-r}}{\overline{f+1}.\overline{f+2}.na^2} - \frac{\overline{g+1}.\overline{g+2}.\mathcal{Q}^{2-r}}{\overline{f+1}.\overline{f+2}.\overline{f+3}.na^3}\,(r)}$$

$$+ \frac{p+m}{m} \times \frac{p+m-1}{m-1} \times \frac{p+m-2}{m-2}\,(r) \times \frac{A}{a^r}.\quad \mathcal{Q}.E.I.$$

COROLLARY.

290. It appears from hence that the Coefficient of A, the given Fluent, will always be equal to *that* of the laſt Term of the preceding Series, multiply'd by $\overline{p+m} \times n$: For, ſeeing the Coefficient of the ſaid laſt Term (whoſe Diſtance from the firſt, incluſive, is denoted by r) muſt be $\frac{\overline{g+1}.\overline{g+2}.\overline{g+3} \ldots\ldots \overline{g+r-1}}{\overline{f+1}.\overline{f+2}.\overline{f+3} \ldots\ldots \overline{f+r}} \times \frac{1}{na^r}$ (by the Law of the Series) where $f+r=m$ and $g+r-1=p+m-1$ (as appears from above) it follows, by inverting the Order of both Progreſſions, that $\frac{\overline{p+m-1}.\overline{p+m-2}.\,(r-1)}{m.\overline{m-1}.\overline{m-2}\,(r)} \times \frac{1}{na^r}$ will alſo expreſs the ſame Coefficient: Which, multiply'd by $\overline{p+m} \times n$, gives $\frac{\overline{p+m}.\overline{p+m-1}.\overline{p+m-2}\,(r)}{m.\overline{m-1}.\overline{m-2}\,(r)}$ $\frac{1}{a^r}$, the very Coeffieient of A, above determined. The Uſe of this Concluſion will be ſeen in what follows.

PRO-

PROB. VII.

291. *The ſame being*, ſtill, *given*; *to find the Fluent of*

$$\overline{a+cz^n}|^m \times z^{pn-vn-1}\dot{z}.$$

By proceeding as in the laſt Problem, the required Fluent of $\overline{a+cz^n}|^m \times z^{pn-vn-1}$ is derived from *that* of $\overline{a+cz^n}|^m \times z^{pn+vn-1}\dot{z}$ (*given by Prob.* 4.) and comes out

$$= \mathcal{Q}^{m+1} z^{pn} \times \overline{\frac{z^{-vn}}{\overline{q+1}.na} - \frac{\overline{s+2}.cz^{n-vn}}{\overline{q+1}.\overline{q+2}.na^2} + \frac{\overline{s+2}.\overline{s+3}.c^2z^{2n-vn}}{\overline{q+1}.\overline{q+2}.\overline{q+3}.na^3}\,(v)} \pm \frac{t-1}{p-1} \times \frac{t-2}{p-2} \times \frac{t-3}{p-3}\,(v) \times \frac{c^v A}{a^v}:$$

Where, $\mathcal{Q}=a+cz^n$, $q=p-v-1$, $s=m+q$, $t=p+m+1$: And where, the Coefficient of A is equal to that of the laſt of the preceding Terms, multiply'd by $-\overline{m+p} \times nc$. If the Manner of deducing the required Fluent, in this, and the laſt, Problem, ſhould not appear ſufficiently plain and ſatisfactory to the Beginner; the ſame Concluſions may be, otherwiſe, brought out; by finding A, in Terms of B, C, or D, from the ſeveral particular Equations in *Art.* 283. or, by aſſuming a deſcending Series, inſtead of an aſcending one. *Vid. Art.* 284.

PROB. VIII.

292. *The ſame being*, ſtill, *given*; *to find the Fluent of*

$$\overline{a+cz^n}|^{m+r} \times z^{pn+vn-1}\dot{z}.$$

Let the Fluent of $\overline{a+cz^n}|^m \times z^{pn+vn-1}\dot{z}$ (*given by Prob.* 4.) be denoted by B, and that required by F:

Then, if $p+v$ be put $=\acute{p}$, the Value of F (the Fluent of $\overline{a+cz^n}^{\,m+r}\times z^{\acute{p}n-1}\dot{z}$) will be given from that of B (the Fluent of $\overline{a+cz^n}^{\,m}\times z^{\acute{p}n-1}\dot{z}$) by writing B for A and $\acute{p}$ for p, in *Art.* 287. Whence we get $F=\mathcal{Q}^{m+1}$ $z^{\acute{p}n}\times\overline{\frac{\mathcal{Q}^{r-1}}{gn}+\frac{fa\mathcal{Q}^{r-2}}{g.\overline{g-1}.n}+\frac{f.\overline{f-1}\,a^2\mathcal{Q}^{r-3}}{g.\overline{g-1}.\overline{g-2}.n}\ (r)}+\frac{m+1}{\acute{p}+m+1}\times\frac{m+2}{\acute{p}+m+2}\times\frac{m+3}{\acute{p}+m+3}\ (r)\times a^rB$: Where $\acute{p}=p+v$, $f=m+r$, $g\ (=\acute{p}+m+r)=p+m+v+r$, and $\mathcal{Q}=a+cz^n$.

Which Fluent, by ſubſtituting the Value of B (*in Prob.* 4.) becomes $F=\mathcal{Q}^{m+1}z^{\acute{p}n}\times\overline{\frac{\mathcal{Q}^{r-1}}{gn}+\frac{fa\mathcal{Q}^{r-2}}{g.\overline{g-1}.n}}$ $\overline{+\frac{f.\overline{f-1}.a^2\mathcal{Q}^{r-3}}{g.\overline{g-1}.\overline{g-2}.n}\ (r)}+\frac{m+1}{\acute{p}+m+1}\times\frac{m+2}{\acute{p}+m+2}\ (r)$ $\times a^r\times\mathcal{Q}^{m+1}z^{pn}\times\overline{\frac{z^{vn-n}}{\overline{s+1}.nc}-\frac{qaz^{vn-2n}}{\overline{s+1}.snc^2}+\frac{q.\overline{q-1}.a^2z^{vn-3n}}{\overline{s+1}.s.\overline{s-1}.nc^3}}$ $(v)\pm\frac{m+1}{\acute{p}+m+1}\times\frac{m+2}{\acute{p}+m+2}\ (r)\times a^r\times\frac{p}{t}\times\frac{p+1}{t+1}$ $(v)\times\frac{a^vA}{c^v}$: Where $q=p+v-1$, $s=m+q=m+p+v-1$, and $t=p+m+1$; and where the Sign of the laſt Term is + or — according as v is an even or odd Number. *Q. E. I.*

292. If the laſt Term of the firſt Series, excluſive of the general Multiplicator $\mathcal{Q}^{m+1}z^{\acute{p}n}$, be donoted by β, the Multiplicator, $\frac{m+1}{\acute{p}+m+1} \times \frac{m+2}{\acute{p}+m+2}(r) \times a^r$, to the ſecond Series will be $= \overline{m+1} \times na\beta$ *; and therefore the firſt Term of this Series, including its Multiplicators, is $= \frac{\overline{m+1}.a\beta\mathcal{Q}^{m+1}z^{pn+vn}}{\overline{s+1}.cz^n}$: Which, if R be put to denote the laſt Term $\beta\mathcal{Q}^{m+1}z^{pn+vn}$ of the firſt Series (with its Multiplicator) will be expounded by $\frac{\overline{m+1}.aR}{\overline{s+1}.cz^n}$. Hence it follows, that the Fluent of $\overline{a+cz^n}^{m+r} \times z^{pn+vn-1}\dot{z}$, given above, will alſo be truly expreſſed by

$$\frac{\mathcal{Q}^{m+r} \times z^{pn+vn}}{gn} + \frac{f}{g-1} \times \frac{aH}{\mathcal{Q}} + \frac{f-1}{g-2} \times \frac{aI}{\mathcal{Q}} + \frac{f-2}{g-3} \times \frac{aK}{\mathcal{Q}}\,(r) + \frac{m+1}{s+1} \times \frac{aR}{cz^n} - \frac{q}{s} \times \frac{aS}{cz^n} - \frac{q-1}{s-1} \times \frac{aT}{cz^n} - \frac{q-2}{s-2} \times \frac{aV}{cz^n}\,(v) \pm \frac{\overline{m+1}.\overline{m+2}.\overline{m+3}\,(r) \times p.\overline{p+1}.\overline{p+2}\,(v)}{\overline{\acute{p}+m+1}.\overline{\acute{p}+m+2}\,(r) \times t.\overline{t+1}.\overline{t+2}\,(v)} \times \frac{a^{v+r}A}{c^v}:$$

Where H, I, K, L . . . . . R, S, T, V, *&c.* repreſent the Terms immediately preceding thoſe where they ſtand, under their proper Signs: R being the laſt Term of the firſt Series; alſo $f=m+r$, $g=m+r+p+v$, $q=p+v-1$, $s=m+q$, $t=m+p+1$, and $\mathcal{Q}=a+cz^n$.

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* Art. 287.

COROLLARY II.

293. Since the Diviſor, $\overline{\acute{p}+m+1}.\overline{\acute{p}+m+2}\;(r)\times t.\overline{t+1}.\overline{t+2}\;(v)$, of the laſt Term of the Fluent (by ſubſtituting for t and $\acute{p}$ &c.) is $=\overline{p+m+1}.\overline{p+m+2}\;(v)\times\overline{p+v+m+1}.\overline{p+v+m+2}\;(r)$: Where, the laſt Factor $(p+m+v)$ of the firſt Progreſſion, is leſs by Unity than the firſt Factor of the Second; it is evident that the ſaid ſecond Progreſſion is only a Continuation of the firſt to *r more* Factors: And ſo, the laſt Term of the Fluent, where A is found, is truly expreſſed by $\pm$

$$\frac{p.\overline{p+1}.\overline{p+2}\;(v)\times\overline{m+1}.\overline{m+2}.\overline{m+3}\;(r)}{\overline{m+p+1}.\overline{m+p+2}.\overline{m+p+3}\;(v+r)}\times\frac{a^{v+r}A}{c^{v}}.$$

Hence it follows, that the Fluent of $\overline{a+cz^{n}}|^{m+r}\times z^{pn+vn-1}\dot{z}$, or that of $\overline{a-bz^{n}}|^{m+r}\times z^{pn+vn-1}\dot{z}$ (making $c=-b$) will, when $a-bz^{n}$ becomes equal to Nothing, be barely $=$

$$\frac{p.\overline{p+1}.\overline{p+2}\;(v)\times\overline{m+1}.\overline{m+2}.\overline{m+3}\;(r)}{\overline{m+p+1}.\overline{m+p+2}.\overline{m+p+3}\;(v+r)}\times\frac{a^{v+r}A}{b^{v}}:$$

A being the Fluent of $\overline{a-bz^{n}}|^{m}\times z^{pn-1}\dot{z}$, in that Circumſtance, v and r whole poſitive Numbers, and p and $m+1$ any poſitive Numbers, either whole or broken.

SCHOLIUM.

294. If the Fluent of $\overline{a+cz^{n}}|^{m+r}\times z^{pn-1}\dot{z}$ (given by *Prob.* 5.) be denoted by C; then (F) the Fluent of $\overline{a+cz^{n}}|^{\acute{m}}\times z^{pn+vn-1}\dot{z}$ (where $\acute{m}=m+r$) will be had, from C (by *Prob.* 4.) according to a new Form, dif-

ferent from thoſe already given. And, by following the ſame Method, the Fluents of $\overline{a+cz^{n}}|^{m-r} \times z^{pn+vn-1}\dot{z}$, $\overline{a+cz^{n}}|^{m+r} \times z^{pn-vn-1}\dot{z}$, and $\overline{a+cz^{n}}|^{m-r} \times z^{pn-vn-1}\dot{z}$ may alſo be found, each, according to two different Forms, from a Combination of the correſponding Caſes in the foregoing Problems.

But, as it is extremely tireſome to repeat the ſame thing, again and again, where ſuch a Number of Symbols are neceſſarily concerned, I ſhall here put down one Solution to each Caſe (becauſe of their Uſe) leaving the Proceſs and the other Forms (which contain no new Difficulty) to *Thoſe* who will be at the Trouble to ſet about them.

1°. The Fluent of $\overline{a+cz^{n}}|^{m-r} \times z^{pn+vn-1}\dot{z}$ is $=$

$$-\frac{Q^{m-r+1} \times z^{pn+vn}}{\overline{f+1}.na} + \frac{g+1}{f+2} \times \frac{QH}{a} + \frac{g+2}{f+3} \times \frac{QI}{a}\ (r)$$

$$-\frac{QR}{cz^{n}} - \frac{q}{s} \times \frac{aS}{cz^{n}} - \frac{q-1}{s-1} \times \frac{aT}{cz^{n}} - \frac{q-2}{s-2} \times \frac{aV}{cz^{n}}\ (v)$$

$$+\frac{s+1}{m} \times \frac{s}{m-1} \times \frac{s-1}{m-2}\,(r) \times \frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2}\,(v) \times \frac{a^{v-r}A}{c^{v}}.$$

Where H, I, K, L .... R, S, T, &c. denote the Terms immediately preceding thoſe where they ſtand, under their proper Signs; R being the laſt Term of the firſt Series, alſo $Q = a+cz^{n}$, $f = m-r$, $g = p+m+v-r$, $q = p+v-1$, $s = m+p+v-1$, $t = p+m+1$, and $A =$ the given Fluent of $\overline{a+cz^{n}}|^{m} \times z^{pn-1}\dot{z}$.

2°. The Fluent of $\overline{a+cz^{n}}|^{m+r} \times z^{pn-vn-1}\dot{z}$ is =

$$\frac{Q^{m+r+1}\times z^{pn-vn}}{\overline{q+1}.na}-\frac{s+2}{q+2}\times\frac{Hcz^{n}}{a}-\frac{s+3}{q+3}\times\frac{Icz^{n}}{a}\ (v)$$

$$-\frac{Rcz^{n}}{Q}+\frac{f}{g-1}\times\frac{aS}{Q}+\frac{f-1}{g-2}\times\frac{aT}{Q}+\frac{f-2}{g+3}\times\frac{aV}{Q}\ (r)$$

$$+\frac{s+2}{q+1}\times\frac{s+3}{q+2}\times\frac{s+4}{q+3}(v)\times\frac{m+1}{t}\times\frac{m+2}{t+1}\times\frac{m+3}{t+2}(r)\times\frac{\overline{-c}|^{v}A}{a^{v-r}}:$$

Where $q=p-v-1$, $s=m+r+q$, $f=m+r$, $g=p+m+r$, and the reſt as in the preceding Caſe.

3°. The Fluent of $\overline{a+cz^{n}}|^{m-r} \times z^{pn-vn-1}\dot{z}$ is =

$$-\frac{Q^{m-r+1}\times z^{pn-vn}}{\overline{f+1}.na}+\frac{g+1}{f+2}\times\frac{QH}{a}+\frac{g+2}{f+3}\times\frac{QI}{a}(r)$$

$$-\frac{s+1}{q+1}\times\frac{QR}{a}-\frac{s+2}{q+2}\times\frac{Scz^{n}}{a}-\frac{s+3}{q+3}\times\frac{Tcz^{n}}{a}\ (v)$$

$$+\frac{\overline{t-1}.\overline{t-2}.\overline{t-3}.\overline{t-4}.\overline{t-5}\ (r+v)}{m.\overline{m-1}.\overline{m-2}\,(r)\times\overline{p-1}\ \overline{p-2}.\overline{p-3}\ (v)}\times\frac{\overline{-c}|^{v}A}{a^{r+v}}.$$

In which $f=m-r$, $g=m+p-r-v$, $q=p-v-1$, $s=q+m$, and the reſt as before.

295. From what has been delivered in this Section, the Fluents of various Forms of Fluxions may be exhibited, by means of circular Arcs and Logarithms. For, ſince the Fluents of $\overline{a+cz^{n}}|^{-1}\times z^{\frac{1}{2}n-1}\dot{z}$, $\overline{a+cz^{n}}|^{-\frac{1}{2}}\times z^{\frac{1}{2}n-1}\dot{z}$, and $\overline{a+cz^{n}}|^{-\frac{1}{2}}\times z^{-1}\dot{z}$ (which I call original Ones) are all of them explicable by one, or the other, of theſe two Kinds of Quantities (as will appear farther on) *thoſe* of $\overline{a+cz^{n}}|^{-1\pm r}\times z^{\frac{1}{2}n\pm vn-1}\dot{z}$, $\overline{a+cz^{n}}|^{-\frac{1}{2}\pm r}\times z^{\frac{1}{2}n\pm vn-1}\dot{z}$, and $\overline{a+cz^{n}}|^{-\frac{1}{2}\pm r}\times z^{\pm vn-1}\dot{z}$ will alſo be given from thence, by the fore-

going Theorems. Whence the moſt uſeful Forms of Fluents in *Cotes's Harmonia Menſurarum* will be obtained, beſides ſome others, more general than any, of the ſame Kind, put down by that ſagacious Author.

Here follow a few Examples of ſome of the moſt uſeful Caſes.

EXAMPLE I.

296. *Let the Fluxion given be* $\frac{z^{2v}\dot{z}}{\sqrt{d^2\pm z^2}}$ (or $\overline{d^2+z^2}|^{-\frac{1}{2}} \times z^{2v}\dot{z}$) *$v$ being any whole poſitive Number.*

Then, the Fluent of $\overline{d^2\pm z^2}|^{-\frac{1}{2}}\times\dot{z}$, or $\frac{\dot{z}}{\sqrt{d^2\pm z^2}}$ being $=$ *hyp. Log.* $\frac{z+\sqrt{d^2+z^2}}{d}$; or, equal to the Arch whoſe Sine is $\frac{z}{d}$ and Radius Unity*; according as the ſecond Term, in $d^2\pm x^2$, is poſitive or negative; let A be, therefore, taken to denote the ſaid Arch, or Logarithm; and let $\overline{d^2\pm z^2}|^{-\frac{1}{2}}\times\dot{z}$ be compared with $\overline{a+cz^n}|^m\times z^{pn-1}\dot{z}$ (whoſe Fluent is, all along, ſuppoſed to be given $=A$) and you will have $a=d^2$, $c=\pm 1$, $n=2$, $m=-\frac{1}{2}$, $2p-1=0$, and therefore $p=\frac{1}{2}$: Whence, by ſubſtituting thoſe Values in *Art.* 283. we likewiſe get $q\ (p+v-1)=\frac{2v-1}{2}$, $s\ (m+q)=v-1$, $t\ (m+p+1)=1$; and, conſequently, the Fluent ſought

$$=\overline{d^2\pm z^2}|^{\frac{1}{2}}\times\overline{\pm\frac{z^{2v-1}}{2v}-\frac{\overline{2v-1}.d^2z^{2v-3}}{2v.\overline{2v-2}}\pm\frac{\overline{2v-1}.\overline{2v-3}.d^4z^{2v-5}}{2v.\overline{2v-2}.\overline{2v-4}}-\frac{\overline{2v-1}.\overline{2v-3}.\overline{2v-5}.d^6z^{2v-7}}{2v.\overline{2v-2}.\overline{2v-4}.\overline{2v-6}}}$$

(v)

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* Art. 126. 142.

$(v) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8}\ (v) \times d^{2v}$: In which the laſt Term is negative, when the given Fluxion is $\frac{z^{2v}\dot{z}}{\sqrt{d^2+z^2}}$, and v, at the ſame time, an odd Number; but in all other Caſes, affirmative.

EXAMPLE II.

297. *Let* $z^{2v}\dot{z}\sqrt{d^2 \pm z^2}$ (or $\overline{d^2 \pm z^2}|^{-\frac{1}{2}+1} \times z^{2v}\dot{z}$) *be propounded.*

Here, denoting the Fluent of $\overline{d^2 \pm z^2}|^{-\frac{1}{2}}\dot{z}$ by A (as above) and comparing $\overline{d^2 \pm z^2}|^{-\frac{1}{2}+1} \times z^{2v}\dot{z}$, with $\overline{a+cz^n}|^{m+r} \times z^{pn+vn-1}\dot{z}$ (*Vid. Prob.* 8.) we have $r=1$, and the reſt as in the laſt Example: Whence alſo $\acute{p}\ (p+v) = v+\frac{1}{2}$, $f\ (m+r) = \frac{1}{2}$, $g = v+1$, $Q = d^2 \pm z^2$, and the Fluent itſelf $= \frac{z^{2v+1}\sqrt{d^2 \pm z^2}}{2v+2} \pm \frac{d^2R}{2vz^2} \mp \frac{\overline{2v-1}.d^2S}{\overline{2v-2}.z^2} \mp \frac{\overline{2v-3}.d^2T}{\overline{2v-4}.z^2}\ (1+v) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6}\ (v) \times \frac{d^{2v+2}A}{2v+2}$* ($R$, S, T, &c. being the preceding Terms with their Signs) $= \frac{\sqrt{d^2 \pm z^2}}{2v+2} \times \overline{z^{2v+1} \pm \frac{d^2z^{2v-1}}{2v} - \frac{\overline{2v-1}.d^4z^{2v-3}}{2v.\overline{2v-2}} \pm \frac{\overline{2v-1}.\overline{2v-3}.d^6z^{2v-5}}{2v.\overline{2v-2}.\overline{2v-4}}}\ (v+1) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8}\ (v) \times \frac{d^{2v+2}A}{2v+2}$: Where the Sign of the laſt Term muſt be regulated as in the pre-

* Art. 292.

preceding Example—If the Fluent of $\frac{z^{-vn}\dot{z}}{\sqrt{d^2 \pm z^2}}$, or of $z^{-vn}\dot{z}\sqrt{d^2 \pm z^2}$ (in which the Exponent is negative) be required; the Anſwer will be had in finite Terms, independent of *A*, *by Art.* 85.

EXAMPLE III.

298. *Wherein the Fluxion propoſed is* $\overline{d^n - z^n}|^{-\frac{1}{2}+r} \times z^{\frac{1}{2}n+vn-1}\dot{z}$; *r and v being any whole poſitive Numbers,*

Since the Fluent of $\overline{d^n - z^n}|^{-\frac{1}{2}} \times z^{\frac{1}{2}n-1}\dot{z}$ (as will appear hereafter) is truly expreſſed by $\frac{2}{n} \times$ *Arch*, whoſe Sine is $\frac{z^{\frac{1}{2}n}}{d^{\frac{1}{2}n}}$ and Radius Unity, let this Value be denoted by *A*; and then, by writing d^n for a, -1 for c, $-\frac{1}{2}$ for m, and $\frac{1}{2}$ for p, *in Art.* 292. we ſhall have $f\ (m+r) = \frac{2r-1}{2}$, $g\ (m+p+r+v) = r+v$, $q\ (p+v-1) = \frac{2v-1}{2}$, $s\ (m+q) = v-1$, $t\ (p+m+1) = 1$, $\mathcal{Q}\ (a+cz^n) = d^n - z^n$, and the Fluent, itſelf, equal to

$$\frac{\mathcal{Q}^{r-\frac{1}{2}}\,z^{vn+\frac{1}{2}n}}{\overline{r+v}.n} + \frac{\overline{2r-1}}{r+v-1} \times \frac{\frac{1}{2}d^n H}{\mathcal{Q}} + \frac{2r-3}{r+v-2} \times \frac{\frac{1}{2}d^n I}{\mathcal{Q}} + \frac{2r-5}{r+v-3} \times \frac{\frac{1}{2}d^n K}{\mathcal{Q}}\ (r) - \frac{\frac{1}{2}d^n R}{vz^n} + \frac{2v-1}{v-1} \times \frac{\frac{1}{2}d^n S}{z^n} + \frac{2v-3}{v-2} \times \frac{\frac{1}{2}d^n T}{z^n}\ (v) + \frac{1.3.5.7\ (r) \times 1.3.5.7\ (v)}{2.4.6.8.10.12\ (r+v)}$$

*Art. 293. * $\times\, d^{rn+vn}A$: In which *H*, *I*, *K*. . . *R*, *S*, *T*, &c. denote the preceding Terms with their Signs; *R* being the laſt Term of the firſt Series. Hence, becauſe all the Terms, but the laſt, vaniſh, when $\mathcal{Q}=0$, it follows that the whole Fluent of $\overline{d^n-z^n}^{r-\frac{1}{2}} \times z^{vn+\frac{1}{2}n-1}\dot{z}$, generated while z, from Nothing, becomes equal to d, is truly expreſſed by $\frac{1.3.5.7\ (r) \times 1.3.5.7\ (v)}{2.4.6.8.10.12\ (r+v)} \times d^{rn+vn}A$, or by $\frac{1.3.5.7\ (r) \times 1.3.5.7\ (v)}{2.4.6.8.10.12\ (r+v)} \times \frac{d^{rn+vn}G}{n}$; G being the Semi-Periphery of the Circle whoſe Radius is Unity.

EXAMPLE IV.

299. *Let it be required to find the* whole *Fluent of* $\frac{\overline{a-bz^n}^m \times z^{pn-1}\dot{z}}{\overline{d+kz^n}^\beta}$, *generated while* bz^n, *from Nothing, becomes*$=a$; *that of* $\overline{a-bz^n}^m \times z^{pn-1}\dot{z}$ *being given* $(=A.)$

Here, by expanding $\overline{d+kz^n}^{-\beta}$, our given Fluxion becomes $= \overline{a-bz^n}^m \times z^{pn-1}\dot{z}$ into $d^{-\beta} \times \overline{1 -}$ $\overline{\frac{\beta kz^n}{d} + \frac{\beta.\beta+1.k^2z^{2n}}{1.2.d^2} - \frac{\beta.\beta+1.\beta+2.k^3z^{3n}}{1.2.3.d^3}}$ &c.

Which Series being compared with $e+fz^n+gz^{2n}$ &c. (*Vid. Art.* 286.) we have $e=1$, $f=-\frac{\beta k}{d}$, $g=\frac{\beta.\beta+1.k^2}{1.2.d^2}$, &c. and conſequently the Fluent ſought (by ſubſtituting theſe Values) equal to $\frac{A}{d^\beta}$ into $1 -$ $\frac{p}{t} \times \frac{\beta}{1} \times \frac{ak}{bd} + \frac{p}{t}.\frac{p+1}{t+1} \times \frac{\beta}{1}.\frac{\beta+1}{2} \times \overline{\frac{ak}{bd}}^2 -$

$\frac{p}{t}\cdot\frac{p+1}{t+1}\cdot\frac{p+2}{t+2}\times\frac{\beta}{1}\cdot\frac{\beta+1}{2}\cdot\frac{\beta+2}{3}\times\left.\overline{\frac{ak}{bd}}\right|^{3}+$ &c. (t being $=p+m+1$.)

Here the Values of $m+1$, n and p are ſuppoſed poſitive *; and it is requiſite that $1+\frac{ak}{bd}$ ſhould alſo be poſitive; otherwiſe the Fluent will fail. Although the Series brought out above runs on to Infinity, yet it may be ſum'd, in many Caſes: Thus, if the given Fluxion be $\frac{\left.\overline{a-bz^{n}}\right|^{-\frac{1}{2}}\times z^{\frac{1}{2}n-1}\dot{z}}{d+kz^{n}}$; then, the foreſaid Series becoming $1-\frac{1}{2}\times\frac{ak}{bd}+\frac{1}{2}\times\frac{3}{4}\times\left.\overline{\frac{ak}{bd}}\right|^{2}-$ &c. its Sum will be $\left.\overline{1+\frac{ak}{bd}}\right|^{-\frac{1}{2}}$: And conſequently $\frac{A}{d}\times\left.\overline{1+\frac{ak}{bd}}\right|^{-\frac{1}{2}}$ = the Fluent ſought: Where, A (the *whole* Fluent of $\left.\overline{a-bz^{n}}\right|^{-\frac{1}{2}}\times z^{\frac{1}{2}n-1}\dot{z}$) being $=\frac{1}{n\sqrt{b}}\times$ Semi-Periphery of the Circle whoſe Radius is Unity, the Fluent given above will, therefore, be $=\frac{1}{n\sqrt{bd^{2}+adk}}$ × by the ſame Semi-Periphery. If the Reader is deſirous to ſee a further Application of the Summation of Serieſes, to the finding of Fluents, I muſt refer him to my *Diſſertations* (where it is handled in a general Manner) having neither Room nor Inclination to treat of it here.

* Art. 286.

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SECTION IV.

Of the Tranſformation of Fluxions.

301. BY the Transformation of Fluxions may be underſtood, the reducing *any* fluxional Quantity to a different, or more commodious, Form; according to which Senſe, a great Part of the ſecond Section would properly fall under this Head. But, *what* is here propoſed, and *what* is commonly meant by the Transformation of Fluxions, *is*, the Method of ordering thoſe Kinds of Expreſſions which involve one variable Quantity *only* with its Fluxion; which, yet, are ſo affected by radical Signs, that the Fluent, without an Infinite Series, would be impracticable, were it not for a new Subſtitution, or ſome other Kind of Tranſformation, whereby the given Fluxion is render'd more manageable.

Something of this Sort has been already touch'd upon in *Art.* 83. And in what follows I ſhall farther point out and exemplify the principal Caſes wherein ſuch a *Procedure* will be of Service.

302. *If the Number of Dimenſions of the variable Quantity, without the* Vinculum, *increaſed by Unity, be ſome aliquot Part, or Parts, of the Dimenſions of the ſame Quantity, under the* Vinculum, *the Fluxion will be reduced to a better Form by ſubſtituting for that Power of the variable Quantity, which ariſes by dividing its Exponent, under the* Vinculum, *by the Denominator of the Fraction expreſſing the ſaid* aliquot *Part, or Parts.*

Thus, if the Fluxion propounded be $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{c^n \pm z^n}}$; by ſubſtituting $x = z^{\frac{1}{2}n}$, and taking the Fluxion of both Sides of the Equation, we have $\dot{x} = \frac{1}{2}nz^{\frac{1}{2}n-1}\dot{z}$; and therefore $z^{\frac{1}{2}n-1}\dot{z} = \frac{\dot{x}}{\frac{1}{2}n}$: Which Value, with that of z^n, being wrote for their Equals, in the given Fluxion, it will be transformed to $\frac{\dot{x}}{\frac{1}{2}n\sqrt{c^n \pm x^2}}$: Which, putting $a = c^{\frac{1}{2}n}$ (to make the Terms homologous), is alſo expreſſed by $\frac{\dot{x}}{\frac{1}{2}n\sqrt{a^2 \pm x^2}}$: Whereof the Fluent will be given by *Art.* 126. or *Art.* 142. according as the Sign of x^2 is poſitive or negative.

303. If the Power of the variable Quantity under the *Vinculum* has a Coefficient, it will be beſt to bring that Coefficient without the *Vinculum.*

Ex. 2. Where let the Fluxion given be $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{a + cz^n}}$: Which, by bringing c without the *Vinculum*, becomes $\frac{z^{\frac{1}{2}n-1}\dot{z}}{c^{\frac{1}{2}}\sqrt{\frac{a}{c} + z^n}}$: From whence, by putting $x = z^{\frac{1}{2}n}$, and proceeding as above, we get $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{a}{c} + x^2}}$: Whoſe Fluent, *by Art.* 126. is $\frac{1}{\frac{1}{2}nc^{\frac{1}{2}}} \times$ *hyp. Log.* $x + \sqrt{\frac{a}{c} + x^2}$. This, by reſtoring z, becomes $\frac{2}{nc^{\frac{1}{2}}} \times$ *hyp. Log.* $z^{\frac{1}{2}n} + \sqrt{\frac{a}{c} + z^n}$. Which, corrected) by ſuppoſing it $=0$ when $z=0$) gives, at length, $\frac{2}{nc^{\frac{1}{2}}} \times \overline{\textit{hyp. Log. } z^{\frac{1}{2}n} + \sqrt{\frac{a}{c} + z^n} - \textit{hyp. Log. } \sqrt{\frac{a}{c}}} = \frac{2}{nc^{\frac{1}{2}}} \times$ *hyp. Log.* $\sqrt{\frac{cz^n}{a}} + \sqrt{1 + \frac{cz^n}{a}}$ for the true Fluent of the Quantity propoſed.

But, when c is a negative Quantity, this Fluent fails, becauſe the ſquare Root of c is to be extracted. In this Caſe $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{a}{c}+x^2}}$ muſt be transformed to $\frac{\dot{x}}{\frac{1}{2}n\sqrt{-c}\times\sqrt{\frac{a}{-c}-x^2}}$: And then its Fluent (*by Art.* 142.) will be had $=\frac{1}{\frac{1}{2}n\sqrt{-c}}\times$ the *Arch* of a Circle whoſe Radius is Unity, and Right-Sine $=\frac{x}{\sqrt{\frac{a}{-c}}}=\sqrt{\frac{-cz^n}{a}}$.

Ex. 3. Let the given Fluxion be $\frac{\dot{z}}{z\sqrt{a+cz^n}}$. Which, by bringing c without the *Vinculum*, and putting $x=z^{\frac{1}{2}n}$, is transformed to $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}x\sqrt{\frac{a}{c}+x^2}}$: Whereof the Fluent, *by Art.* 126. is $\frac{1}{n\sqrt{a}}\times$ *hyp. Log.* $\frac{\sqrt{\frac{a}{c}}-\sqrt{\frac{a}{c}+x^2}}{\sqrt{\frac{a}{c}}+\sqrt{\frac{a}{c}+x^2}}=\frac{1}{n\sqrt{a}}\times$ *hyp. Log.* $\frac{\sqrt{a}-\sqrt{a+cz^n}}{\sqrt{a}+\sqrt{a+cz^n}}$. But here, when c is poſitive, the Numerator will be negative; in which Caſe it will be proper to change its Signs, and expreſs the Fluent by $\frac{1}{n\sqrt{a}}\times$ *hyp. Log.* $\frac{\sqrt{a+cz^n}-\sqrt{a}}{\sqrt{a+cz^n}+\sqrt{a}}$. That, ſuch an

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an Alteration of the Signs can make no Difference in the Fluxion, is evident from the Nature of Logarithms; becauſe the Fluxion of the *Log.* of $-x$ $\left(= \frac{-\dot{x}}{-x} = \frac{\dot{x}}{x}\right)$ is the ſame with *that* of the *hyp. Log.* of x. It will be proper to obſerve farther, that, inſtead of the Logarithm above derived, any one of the following, equal, Quantities may be taken; *viz. hyp. Log.* $\frac{\overline{\sqrt{a+cz^n}-\sqrt{a}}\,^2}{cz^n}$ (found by multiplying both the Numerator and Denominator of the foreſaid Logarithm by $\sqrt{a+cz^n}-\sqrt{a}$) $= 2\times$ *hyp. Log.* $\frac{\sqrt{a+cz^n}-\sqrt{a}}{\sqrt{cz^n}}$ (by the Nature of Logarithms) $= 2\times$ *hyp. Log.* $\frac{\sqrt{cz^n}}{\sqrt{a+cz^n}+\sqrt{a}}$ (by multiplying, equally, by $\sqrt{a+cz^n}+\sqrt{a}$) But, take which of theſe Forms you will, the Fluent fails when a is negative; becauſe the general Multiplicator $\frac{1}{n\sqrt{a}}$ is then impoſſible. In this Caſe the Fluent of $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}\times x\sqrt{\frac{a}{c}+x^2}}$, or its Equal $\frac{\dot{z}}{z\sqrt{a+cz^n}}$, will be given *by Art.* 142. and is expounded by $\frac{1}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{-a}{c}}}$ $\times A = \frac{2}{n\sqrt{-a}}$; where *A* denotes the *Arch* whoſe Radius is Unity, and Secant $\frac{x}{\sqrt{\frac{-a}{c}}}$ $\left(=\sqrt{\frac{cz^n}{-a}}\right)$.

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In the ſame Manner the Fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{a+cz^n}$, is found $=\frac{1}{n\sqrt{ac}}\times$ *Arch*, whoſe Radius is Unity and Tangent $\sqrt{\frac{cz^n}{a}}$, or equal to $\frac{1}{n\sqrt{-ca}}\times$ *hyp. Log.* $\frac{\sqrt{a}+\sqrt{-cz^n}}{\sqrt{a}-\sqrt{-cz^n}}$, according as the Value of c is affirmative or negative; a being ſuppoſed affirmative.

304. *When the Power, or Powers, of the variable Quantity without the* Vinculum, *or radical Sign, fall, moſtly, in the Denominator, it may be of Uſe to ſubſtitute for the Reciprocal of the ſaid Quantity, or for the Quotient which ariſes by dividing ſome known Quantity, either, by it, or by ſome Compound of it in the Denominator.*

Ex. 1. *Let the propoſed Fluxion be* $\frac{a^3\dot{z}}{z^2\sqrt{a^2+z^2}}$; then, putting $x=\frac{a^2}{z}$, we have $z=\frac{a^2}{x}$, and $\dot{z}=-\frac{a^2\dot{x}}{x^2}$; and conſequently $\frac{a^3\dot{z}}{z^2\sqrt{a^2+z^2}}=\frac{-x\dot{x}}{\sqrt{x^2+a^2}}$: Whereof the Fluent is $-\sqrt{x^2+a^2}=-\sqrt{\frac{a^4}{z^2}+a^2}$.

Ex. 2. *Let the given Fluxion be* $\frac{z\dot{z}}{\overline{a+z}|^3\times\sqrt{a^2+az+z^2}}$.

Here, putting $x=\frac{a^2}{a+z}$, we have $z=\frac{aa-ax}{x}=a\times\frac{a-x}{x}$, $\dot{z}=-\frac{a^2\dot{x}}{x^2}$, $z\dot{z}=-\frac{a^3\dot{x}\times\overline{a-x}}{x^3}$, $\sqrt{a^2+az+z^2}=\frac{a}{x}\sqrt{a^2-ax+x^2}$; and therefore the

Quantity propoſed is transformed to $\frac{x^2\dot{x}-ax\dot{x}}{a^4\sqrt{a^2-ax+x^2}}$: Whoſe Fluent may be found from a Table of Logarithms; as will appear farther on.

305. *If the Fluxion given is affected by two different Surds, and the rational Factor, or the Quantity without the* Vinculum, *be in a conſtant Ratio to the Fluxion of the Quantity under the* Vinculum *of either Surd, or be related to it as in* Art. 83. *the given Fluxion will be reduced to a more ſimple Form, by ſubſtituting for that Surd.*

Ex. 1. *Let* $\frac{z\dot{z}\sqrt{b^2+z^2}}{\sqrt{c^2-z^2}}$ *be propounded.*

Then, putting $x=\sqrt{b^2+z^2}$, we have $z^2=x^2-b^2$, $z\dot{z}=x\dot{x}$, and $\sqrt{c^2-z^2}=\sqrt{c^2+b^2-x^2}=\sqrt{a^2-x^2}$ (by making $a=\sqrt{c^2+b^2}$) Whence $\frac{z\dot{z}\sqrt{b^2+z^2}}{\sqrt{c^2-z^2}}=\frac{x^2\dot{x}}{\sqrt{a^2-x^2}}$ *.

* Art. 279.

Or, if x be put $=\sqrt{c^2-z^2}$ (inſtead of $\sqrt{b^2+z^2}$); then $z^2=c^2-x^2$, $z\dot{z}=-x\dot{x}$, $\sqrt{b^2+z^2}=\sqrt{b^2+c^2-x^2}=\sqrt{a^2-x^2}$; and conſequently $\frac{z\dot{z}\sqrt{bb+zz}}{\sqrt{cc-zz}}=-\dot{x}\sqrt{a^2-x^2}$: Whoſe Fluent is given by *Art.* 297. or 131.

Ex. 2. *Let the given Fluxion be* $\overline{a+cz^n}\big|^{m}\times\overline{e+fz^n}\big|^{r}\times z^{pn-1}\dot{z}$; *ſuppoſing p to denote any whole poſitive Number* *.

* Art 83.

In this Caſe, let that of the two Quantities, $a+cz^n$ and $e+fz^n$, whoſe Index (m or r) is the moſt complex (which we will ſuppoſe the latter) be put $=x$; then we ſhall have $z^n=\frac{x-e}{f}$; $z^{n-1}\dot{z}=\frac{\dot{x}}{nf}$;

$z^{pn-1}\dot{z}\ (=z^{pn-n}\times z^{n-1}\dot{z}) = \frac{\overline{x-e}^{\,p-1}}{f^{p-1}} \times \frac{\dot{x}}{nf}$; $a+cz^n = a+\frac{cx-ce}{f} = d+\frac{cx}{f}$ (by putting $d = a - \frac{ce}{f}$) and conſequently $\overline{d+\frac{cx}{f}}^{\,m} \times \frac{\overline{x-e}^{\,p-1}\times x^r\dot{x}}{nf^p}$ = the Fluxion propoſed: Where, $p-1$ being a whole poſitive Number, the Value of $\overline{x-e}^{\,p-1}$ will therefore be expreſſed in finite Terms; whence, if m be alſo a whole poſitive Number, the Fluent itſelf will be had in finite Terms: But, if m and r be the Halves of odd Numbers, then the Fluent will be found *(from Art.* 298 or 294.) by means of circular Arcs and Logarithms.

306. *If the given Expreſſion be affected by two Surds wherein the Powers of the variable Quantity are the ſame, and the rational Quantity, without the* Vinculums, *be related to the Fluxion of either Surd, as in Art.* 83. *it may be of Uſe to ſubſtitute for the Quotient, or Ratio, of the two Quantities under the radical Signs; eſpecially, if the Sum of the ſaid radical Signs, or Exponents (ſuppoſing both Surds to be reduced to the Denominator) is a whole Number.*

Ex. 1. *Let the given Fluxion be* $\frac{z^2\dot{z}}{\overline{b^3+z^3}^{\,\frac{2}{3}}\times\overline{c^3-z^3}^{\,\frac{4}{3}}}$.

Then, writing $x = \frac{b^3+z^3}{c^3-x^3}$, we have $z^3 = \frac{c^3x-b^3}{1+x}$; $3z^2\dot{z} = \frac{\overline{c^3+b^3}\times\dot{x}}{\overline{1+x}^{\,2}}$; $\overline{b^3+z^3}^{\,\frac{2}{3}}\times\overline{c^3-z^3}^{\,\frac{4}{3}}\ \left(=\overline{\frac{b^3+z^3}{c^3-z^3}}^{\,\frac{2}{3}}\times\overline{c^3-z^3}^{\,2} = x^{\frac{2}{3}}\times\overline{c^3-\frac{c^3x-b^3}{1+x}}^{\,2}\right) = \frac{\overline{b^3+c^3}^{\,2}\times x^{\frac{2}{3}}}{\overline{1+x}^{\,2}}$;

and conſequently $\frac{z^2\dot{z}}{\overline{b^3+z^3}|^{\frac{2}{3}}\times\overline{c^3-z^3}|^{\frac{4}{3}}}=\frac{x^{-\frac{2}{3}}\dot{x}}{3\times b^3+c^3}$:

Whoſe Fluent is $\frac{x^{\frac{1}{3}}}{b^3+c^3}=\frac{1}{b^3+c^3}\times\sqrt[3]{\frac{b^3+z^3}{c^3-z^3}}$.

Ex. 2. *Let there be given* $\frac{z^{pn-1}\dot{z}}{\overline{a+cz^n}|^{m}\times\overline{e+fz^n}|^{r}}$

Here, putting $x=\frac{e+fz^n}{a+cz^n}$, you will have $z^n=\frac{ax-e}{f-cx}$; $nz^{n-1}\dot{z}=\frac{\overline{af-ce}\times\dot{x}}{\overline{f-cx}|^{2}}$; $z^{pn-1}\dot{z}$ $(=z^{pn-n}\times z^{n-1}\dot{z})=\overline{\frac{ax-e}{f-cx}}|^{p-1}\times\frac{\overline{af-ce}\times\dot{x}}{n\times\overline{f-cx}|^{2}}$; $\overline{a+cz^n}|^{m}\times\overline{e+fz^n}|^{r}$ $(=\overline{a+cz^n}|^{m+r}\times\frac{\overline{e+fz^n}|^{r}}{\overline{a+cz^n}|^{r}}=\overline{a+cx\frac{ax-e}{f-cx}}|^{r+m}\times x^r)=\overline{\frac{af-ce}{f-cx}}|^{m+r}\times x^r$; and conſequently the Fluxion given $=\frac{\overline{ax-e}|^{p-1}\times\overline{f-cx}|^{m+r-p-1}\times x^{-r}\dot{x}}{n\times\overline{af-ce}|^{m+r-1}}$.

Where, if $m+r$ be a whole poſitive Number, greater than p (alſo a whole poſitive Number) the Fluent will be truly had in finite Terms; becauſe both the Serieſes for the Values of $\overline{ax-e}|^{p-1}$ and $\overline{f-cx}|^{m+r-p-1}$ do in that Caſe terminate *. But, if r and $m+r-p-1$ be the Halves of whole Numbers, poſitive or negative, then the Fluent will be given by the laſt Section.

* Art. 99.

307. *A Trinomial is reduced to a Binomial by taking away its middle Term; that is, by ſubſtituting for the Sum or Difference of the Power of the variable Quantity*

in that Term and half its Coefficient; according as the Signs of the two Terms, where the ſaid Quantity is found, are like, or unlike.

Ex. 1. *Let the given Fluxion be* $\dfrac{\dot{z}}{\sqrt{b^2+cz+z^2}}$; then, putting $x=z+\frac{1}{2}c$, or $z=x-\frac{1}{2}c$, we have $\dot{z}=\dot{x}$, and $\sqrt{b^2+cz+z^2}$ $(=\sqrt{b^2+cx-\frac{1}{2}c^2+x^2-cx+\frac{1}{4}c^2})$ $=\sqrt{b^2-\frac{1}{4}c^2+x^2}$; whence (making $a^2=b^2-\frac{1}{4}c^2$) there reſults $\dfrac{\dot{z}}{\sqrt{b^2+cz+z^2}}=\dfrac{\dot{x}}{\sqrt{a^2+x^2}}$: Whoſe Fluent is given, *by Art.* 126.

Ex. 2. *Let the Fluxion given be* $\dfrac{fz^{n-1}\dot{z}}{\sqrt{a+bz^n+cz^{2n}}}$.

Firſt, by bringing c without the *Vinculum*, according to *Art.* 303. we have $\sqrt{a+bz^n+cz^{2n}}=\sqrt{c}\times\sqrt{\dfrac{a}{c}+\dfrac{bz^n}{c}+z^{2n}}$: And, by putting $x=z^n+\dfrac{b}{2c}$, or $z^n=x-\dfrac{b}{2c}$, we alſo get $z^{n-1}\dot{z}=\dfrac{\dot{x}}{n}$, and $\sqrt{\dfrac{a}{c}+\dfrac{bz^n}{c}+z^{2n}}$ $\left(=\sqrt{\dfrac{a}{c}+\dfrac{bx}{c}-\dfrac{bb}{2cc}+x^2-\dfrac{bx}{c}+\dfrac{bb}{4cc}}\right)=\sqrt{\dfrac{a}{c}-\dfrac{bb}{4cc}+x^2}$: Therefore the Fluxion, transformed, is $\dfrac{f\dot{x}}{n\sqrt{c}\times\sqrt{\dfrac{a}{c}-\dfrac{bb}{4cc}+x^2}}$: Whoſe Fluent is given by *Art.* 126. when c is a poſitive Quantity: But, when c is negative, the Fluxion muſt be expreſſed thus, $\dfrac{f\dot{x}}{n\sqrt{-c}\times\sqrt{\dfrac{a}{-c}+\dfrac{bb}{4cc}-x^2}}$: Anſwering to *Form* 2. *Art.* 142.

Ex. 3. *Let* $\frac{fz^{n-1}\dot{z}+gz^{2n-1}\dot{z}+hz^{3n-1}\dot{z}+kz^{4n-1}\dot{z}.}{\overline{a+bz^{n}+cz^{2n}}|^{m}}$ *be propoſed.*

Then, following the Steps of the laſt Example, $\overline{a+bz^{n}+cz^{2n}}|^{m}\left(=c^{m}\times\overline{\frac{a}{c}+\frac{bz^{n}}{c}+z^{2n}}|^{m}\right)$ will be transformed to $c^{m}\times\overline{\frac{a}{c}-\frac{bb}{4cc}+x^{2}}|^{m}$: Moreover, z^{n} being $=x-\frac{b}{2c}=x-d$ (by putting $d=\frac{b}{2c}$) and $z^{n-1}\dot{z}=\frac{\dot{x}}{n}$, we alſo have $z^{2n-1}\dot{z}$ $\left(=z^{n}\times z^{n-1}\dot{z}=\overline{x-d}\times\frac{\dot{x}}{n}\right)=\frac{x\dot{x}-d\dot{x}}{n}$; $z^{3n-1}\dot{z}$ $(=z^{2n}\times z^{n-1}\dot{z})=\overline{x-d}|^{2}\times\frac{\dot{x}}{n}=\frac{x^{2}\dot{x}-2dx\dot{x}+d^{2}\dot{x}}{n}$; &c. &c. From whence, by ſubſtituting theſe ſeveral Values in the given Fluxion, and putting $\frac{a}{c}-\frac{bb}{4cc}=e^{2}$, there comes out

$$\frac{f\dot{x}+g\times\overline{x\dot{x}-d\dot{x}}+h\times\overline{x^{2}\dot{x}-2dx\dot{x}+d^{2}\dot{x}}+\&c.}{nc^{m}\times\overline{ee+xx}|^{m}}:$$

Whoſe Fluent, when the Exponent m is the Half of any *Integer*, poſitive or negative, will be found, by means of circular Arcs and Logarithms, from *Art.* 295.

308. *When the Denominator is a rational Trinomial, or Multinomial (that is, when it is without a* Vinculum) *the beſt Way of proceeding, for the general Part, is, to reſolve the given Fraction into binomial Ones. In order to this, let its Denominator be feigned* $=0$; *by means*

means of which Equation, whoſe Roots muſt be found, you will, by ſubſtracting each Root from the indeterminate Quantity (x), have the binomial Denominators of the required Fractions into which the given One *may be reſolved: Whoſe correſponding Numerators, let be denoted* $A\dot{x}$, $B\dot{x}$, $C\dot{x}$ &c. *then, by putting the Sum of the Fractions, thus ariſing, equal to the given Fraction, and reduciug the whole Equation to the ſame Denominator, the aſſumed Quantities A, B, C &c. by comparing the homologous Terms, will be determined.*

Ex. 1. *Let the given Fraction be* $\frac{\dot{x}}{x^2+ax+b}$; then, feigning $x^2+ax+b=0$, the two Roots of the Equation will be $-\frac{1}{2}a-\sqrt{\frac{1}{4}a^2-b}$, and $-\frac{1}{2}a+\sqrt{\frac{1}{4}a^2-b}$: Which being denoted by p and q, we have $x-p$ and $x-q$ for the two binomial Factors whereby x^2+ax+b may be reſolved, or by whoſe Multiplication ($\overline{x-p}\times\overline{x-q}$) the ſaid Quantity is produced.

Let therefore $\frac{A\dot{x}}{x-p}+\frac{B\dot{x}}{x-q}$ be now aſſumed $\left(=\frac{\dot{x}}{x^2+ax+b}\right)=\frac{\dot{x}}{\overline{x-p}\times\overline{x-q}}$; then, by reducing the whole Equation to one Denomination &c. we get $\overline{A+B}\times x\dot{x}-\overline{qA+pB+1}\times\dot{x}=0$: Whence A is found $=\frac{1}{p-q}$, $B=\frac{1}{q-p}$; and, conſequently,

$$\frac{\dot{x}}{\overline{p-q}\times\overline{x-p}}+\frac{\dot{x}}{\overline{q-p}\times\overline{x-q}}=\frac{\dot{x}}{x^2+ax+b}.$$

Ex. 2. *Let the Quantity propoſed be* $\frac{x^2\dot{x}}{x^3+ax^2+bx+c}$.

Here, if the binomial Factors whereby x^3+ax^2+bx+c is produced be repreſented by $x-p$, $x-q$, and $x-r$, and there be aſſumed $\frac{A\dot{x}}{x-p}+\frac{B\dot{x}}{x-q}+\frac{C\dot{x}}{x-r}$

$\left(=\frac{x^2\dot{x}}{x^3+ax^2+bx+c}\right)=\frac{x^2\dot{x}}{\overline{x-p}\times\overline{x-q}\times\overline{x-r}}$; then, in this Caſe, we ſhall have $A\times\overline{x-q}\times\overline{x-r}+B\times\overline{x-p}\times\overline{x-r}+C\times\overline{x-p}\times\overline{x-q}-x^2=0$; that is, by Reduction,

$$\left.\begin{matrix}A\\B\\C\\-1\end{matrix}\right\}\times x^2-\left.\begin{matrix}\overline{q+r}\times A\\\overline{p+r}\times B\\\overline{p+q}\times C\end{matrix}\right\}\times x+\left.\begin{matrix}qrA\\prB\\pqC\end{matrix}\right\}=0.$$

Whence $A+B+C=1$, $A\times\overline{q+r}+B\times\overline{p+r}+C\times\overline{p+q}=0$, and $Aqr+Bpr+Cpq=0$. Now, from the firſt of theſe Equations, multiply'd by $p+q$, ſubtract the ſecond, and you will have $A\times\overline{p-r}+B\times\overline{q-r}=p+q$: Alſo, from the firſt, multiply'd by pq, ſubtract the third; then $A\times\overline{pq-rq}+B\times\overline{pq-pr}=pq$: Laſtly, from the former of the two Equations thus ariſing, multiply'd by p, ſubtract the latter, then $A\times\overline{pp-pr-pq+qr}=pp$, that is, $A\times\overline{p-q}\times\overline{p-r}=p^2$; and conſequently $A=\frac{p^2}{\overline{p-q}\times\overline{p-r}}$: Whence, by the very ſame Argument, $B=\frac{q^2}{\overline{q-p}\times\overline{q-r}}$, and $C=\frac{r^2}{\overline{r-p}\times\overline{r-q}}$.

309. After the ſame Manner you may proceed in other Caſes: But there is an Artifice, or Compendium, for more readily determining the aſſumed Quantities A, B, C *&c.* by which a great deal of Trouble is avoided: And that is, by conſidering the Equation in ſuch Circumſtances of the indeterminate Quantity x, when it becomes moſt ſimple, or when moſt of its Terms vaniſh.

Thus, in the preceding Example, becauſe $A\times\overline{x-q}\times\overline{x-r}+B\times\overline{x-p}\times\overline{x-r}+C\times\overline{x-p}\times\overline{x-q}-x^2$ is $=0$ (in all Circumſtances of x whatever) let x be taken $=p$; then, all the Terms vaniſhing, except the firſt and laſt,

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have $A\times\overline{p-q}\times\overline{p-r}-p^2=0$; and conſequently $A=\frac{p^2}{\overline{p-q}\times\overline{p-r}}$; *the very ſame as before.*

More univerſally, let the given Fraction be

$$\frac{x^m\dot{x}}{x^n+ax^{n-1}+bx^{n-2}+cx^{n-3}\ \text{\&c.}}=$$

$\frac{x^m\dot{x}}{\overline{x-p}\times\overline{x-q}\times\overline{x-r}\times\overline{x-s}\ \text{\&c.}}$ (where m and n may repreſent any whole poſitive Numbers whatever, provided the latter be greater than the former.) Then, aſſuming $\frac{A\dot{x}}{x-p}+\frac{B\dot{x}}{x-q}+\frac{C\dot{x}}{x-r}+\frac{D\dot{x}}{x-s}$ &c. $=$

$\frac{x^m\dot{x}}{x^n+ax^{n-1}+bx^{n-2}\ \text{\&c.}}$ &c. we ſhall have $A\times\overline{x-q}\times\overline{x-r}\times\overline{x-s}$ &c. $+B\times\overline{x-p}\times\overline{x-r}\times\overline{x-s}$ &c. $+C\times\overline{x-p}\times\overline{x-q}\times\overline{x-s}$ &c. &c. $-x^m=0$: From whence, by expounding x by p, q, r &c. ſucceſſively, we obtain $A=\frac{p^m}{\overline{p-q}\cdot\overline{p-r}\cdot\overline{p-s}\ \text{\&c.}}$, $B=$ $\frac{q^m}{\overline{q-p}\cdot\overline{q-r}\cdot\overline{q-s}\ \text{\&c.}}$, $C=\frac{r^m}{\overline{r-p}\cdot\overline{r-q}\cdot\overline{r-s}\ \text{\&c.}}$, &c. Whence the Fractions themſelves, whereof theſe Quantities are the Coefficients, or Numerators, will likewiſe be given.

But the Numerators thus found may, ſometimes, be more commodiouſly expreſſed by Help of the given Coefficients a, b, c, d &c. ſo as to involve only one of the Roots p, q, r &c. in each Fraction. For, ſince $\overline{x-p}\times\overline{x-q}\times\overline{x-r}$ &c. is ſuppoſed, *univerſally*, $=x^n+ax^{n-1}+bx^{n-2}+cx^{n-3}$ &c. if both Sides of the

Equation be divided by $x-p$, we ſhall have $\overline{x-q} \times \overline{x-r} \times \overline{x-s}$ &c. $= \frac{x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} \text{ \&c.}}{x-p}$ Which laſt Expreſſion, when x is $= p$, that is, when both the Numerator and the Denominator become equal to Nothing, will, manifeſtly, be equal to $(\overline{p-q} \times \overline{p-r} \times \overline{p-s}$ &c.) the Diviſor of A. Therefore, if the Fluxion of the Numerator be taken and divided by that of the Denominator, and p be wrote inſtead of x (*vid. Page* 155.) we ſhall have $np^{n-1} + \overline{n-1} \times ap^{n-2} + \overline{n-2} \times bp^{n-3}$ &c. $= \overline{p-q} \times \overline{p-r} \times \overline{p-s}$ &c. and therefore A $\left(= \frac{p^m}{\overline{p-q} \cdot \overline{p-r} \cdot \overline{p-s} \text{ \&c.}}\right) =$

$$\frac{p^m}{np^{n-1} + \overline{n-1} \cdot ap^{n-2} + \overline{n-2} \cdot bp^{n-3} \text{ \&c.}}.$$

By the very ſame Reaſoning $B =$

$$\frac{q^m}{nq^{n-1} + \overline{n-1} \cdot aq^{n-2} + \overline{n-2} \cdot bq^{n-3} \text{ \&c.}} \quad C =$$

$$\frac{r^m}{nr^{n-1} + \overline{n-1} \cdot ar^{n-2} + \overline{n-2} \cdot br^{n-3} \text{ \&c.}} \text{ \&c.}$$

Hence it appears, that, if the Numerator of the given Fraction be divided by the Fluxion of the Denominator (neglecting $\dot{x}$) and the ſeveral Roots p, q, r &c. (found by feigning the Denominator $= 0$) be, ſucceſſively, ſubſtituted in the Quotient, inſtead of x; I ſay, it is evident, that the Quantities ſo reſulting, divided by $x-p$, $x-q$, $x-r$ &c. will be the required, binomial, Fractions into which the propoſed multinomial One may be reſolved.

310. If ſome of the Roots p, q, r &c. are impoſſible, which is often the Caſe, the Fractions thus found, where the impoſſible Roots are concerned, muſt

be united in Pairs, and ſo reduced to trinomial Ones, in order to take away the *imaginary Terms.*

Thus, let the Fraction propoſed be $\frac{x\dot{x}}{x^3+ax^2+bx+c}$, and let two of the Roots, p and q, of the Equation $x^3+ax^2+bx+c=0$ be impoſſible: Then, $\frac{A\dot{x}}{x-p}+\frac{B\dot{x}}{x-q}+\frac{C\dot{x}}{x-r}$ being $=\frac{x\dot{x}}{x^3+ax^2+bx+c}$, we ſhall, by uniting the *imaginary Terms,* have $\frac{\overline{A+B}\times x\dot{x}-\overline{Aq+Bp}\times\dot{x}}{x^2-\overline{p+q}\times x+pq}+\frac{C\dot{x}}{x-r}$, alſo, $=\frac{x\dot{x}}{x^3+ax^2+bx+c}$; where the impoſſible Quantities deſtroy one another. But, to render this more obvious, let a be taken $=0$, $b=0$, and $c=-1$, ſo that the given Fraction may become $\frac{x\dot{x}}{x^3-1}$; then the three Roots (p, q, r) of the Equation, $x^3-1=0$, will here be $-\frac{1}{2}+\sqrt{\frac{-3}{4}}$, $-\frac{1}{2}-\sqrt{\frac{-3}{4}}$, and 1; whereof the two former are impoſſible. Moreover, by dividing the Numerator (x) by the Fluxion of the Denominator $(3x^2)$ (according to the *Preſcript)* we have $\frac{1}{3x}$; which, by writing p, q, r ſucceſſively, inſtead of x, becomes $\frac{1}{3p}$, $\frac{1}{3q}$ and $\frac{1}{3r}$ for the Values of A, B, and C, reſpectively. Whence $\frac{\overline{A+B}\times x-Aq-Bp}{x^2-\overline{p+q}\times x+pq}+\frac{C}{x-r}\left(=\frac{x}{x^3-1}\right)$ is $=\frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}+\frac{\frac{1}{3}}{x-1}=\frac{1-x}{3x^2+3x+3}+\frac{1}{3x-3}$. But the ſame may be, otherwiſe

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wiſe, inveſtigated, in a more general Manner; by aſſuming $\frac{Px+Q}{x^2+x+1}+\frac{R}{x-1}=\frac{x}{x^3-1}$, and proceeding as in the firſt and ſecond Examples; whence the very ſame Concluſion will be derived.

If the Fraction propoſed be of this Form, *viz.* $\frac{z^{pn-1}\dot{z}}{z^{mn}+az^{mn-n}+bz^{mn-2n}\ \&c.}$, the Method of Reſolution will, *ſtill*, be the ſame: Since, by putting $x=z^n$, the given Expreſſion is reduced to

$$\frac{\frac{1}{n}\times x^{p-1}\dot{x}}{x^m+ax^{m-1}+bx^{m-2}\ \&c.}.$$

It may alſo be proper to obſerve, *that*, in very complicated Caſes, the Application of two, or more, of the ſix foregoing Rules, may become neceſſary. Thus, for Example, if the Fluxion given be $\frac{z^{pn-1}\dot{z}}{\overline{a+cz^n}^{\,m}\times\overline{e+fz^n+gz^{2n}}}$; by reſolving $\frac{1}{e+fz^n+gz^{2n}}$ into two Binomial Fractions, $\frac{A}{h+z^n}+\frac{B}{k+z^n}$ (*according to Art.* 308.) we ſhall have $\frac{z^{pn-1}\dot{z}}{\overline{a+cz^n}^{\,m}\times\overline{e+fz^n+gz^{2n}}}$

$$=\frac{Az^{pn-1}\dot{z}}{\overline{a+cz^n}^{\,m}\times\overline{h+z^n}}+\frac{Bz^{pn-1}\dot{z}}{\overline{a+cz^n}^{\,m}\times\overline{k+z^n}}$$

:Where, if m be a whole poſitive Number, greater than p, the Fluent will be had in finite Terms (*by Art.* 306. *Ex.* 2.)

SECTION V.

The Inveſtigation of Fluents of rational Fractions, of ſeveral Dimenſions, according to the Forms *in* Cotes's HARMONIA MENSURARUM.

311. AS the Subject here propoſed is a Matter of conſiderable Difficulty, and has exercis'd the Attention of ſome of the moſt celebrated Mathematicians (who, yet, ſeem to have condeſcended very little to the Information of their leſs experienc'd Readers) I ſhall endeavour to ſet it in the cleareſt Light poſſible: In order to which, it will be requiſite to premiſe the following Lemmas.

LEMMA I.

If the Sine of the Mean of three equi-different Arcs, ſuppoſing Radius Unity, be multiply'd by the Double of the Co-ſine of the common Difference, and from the Product, the Sine of the leſſer Extreme be ſubtracted, the Remainder will be the Sine of the greater Extreme.

LEMMA II.

312. *If G be taken to denote the greater, and L the leſſer, of two unequal Arcs, and their Difference be expreſſed by D; then will,*

1. $$\frac{Sin. G. \times Co\text{-}ſ. D - Sin. L. \times Rad.}{Sin. D} = Co\text{-}ſ. G$$

2. $$\frac{Co\text{-}ſ. L \times Rad. - Co\text{-}ſ. G \times Co\text{-}ſ. D}{Sin. D} = Sin. G$$

3. $$\frac{Sin. G. \times Rad. - Sin. L \times Co\text{-}ſ. D}{Sin. D} = Co\text{-}ſ. L.$$

The former of theſe two Lemmas may be met with in moſt Authors upon Trigonometry; and the latter is nothing more than a Corollary to the *common* Theorems for finding the Sine and Co-ſine of the Sum and Difference of two given Arcs; for which Reaſons I ſhall not ſtop here to give their Demonſtration.

COROLLARY.

313. If any Arch of the Circle, whoſe Radius is Unity, be denoted by $\mathcal{Q}$, its Sine by s, and its Co-ſine by a; and there be taken $A=2a$, $B=2aA-1$. $C=2aB-A$, $D=2aC-B$, $E=2aD-C$, $F=2aE-D$, *&c.* it follows (*from Lemma* 1.) that,

$Sin.\ 2\mathcal{Q}\ (Sin.\ \mathcal{Q}\times 2a - Sin.\ 0) = 2sa - 0 = sA$
$Sin.\ 3\mathcal{Q}\ (Sin.\ 2\mathcal{Q}\times 2a - Sin.\ \mathcal{Q}) = 2sAa - s = sB$
$Sin.\ 4\mathcal{Q}\ (Sin.\ 3\mathcal{Q}\times 2a - Sin.\ 2\mathcal{Q}) = 2sBa - sA = sC$
$Sin.\ 5\mathcal{Q}\ (Sin.\ 4\mathcal{Q}\times 2a - Sin.\ 3\mathcal{Q}) = 2sCa - sB = sD$
$Sin.\ 6\mathcal{Q}\ (Sin.\ 5\mathcal{Q}\times 2a - Sin.\ 4\mathcal{Q}) = 2sDa - sC = sE$
&c. *&c.*

LEMMA III.

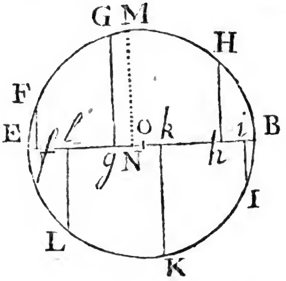
314. *To reſolve the Trinomial* $r^{2n} - 2kr^n x^n + x^{2n}$, *where* n *is any whole Number, into ſimple trinomial Factors.*

Since the firſt Term of the given Quantity $r^{2n} - 2kr^n x^n + x^{2n}$ is diviſible, only, by the Powers of r, and the laſt, only, by thoſe of x; and it appears that r and x are concerned, exactly, alike; let therefore $r^2 - 2arx + x^2$ (where r and x are, *alſo*, alike concerned) be aſſumed for one of the required trinomial Factors, whereby $r^{2n} - 2kr^n x^n + x^{2n}$ may be reſolved: And let $\overline{r^2 - 2arx + x^2} \times \overline{r^8 + Ar^7x + Br^6x^2 + Cr^5x^3 + Dr^4x^4 + Cr^3x^5 + Br^2x^6 + Arx^7 + x^8}$ (where r and x are, *ſtill*, affected alike) be aſſumed $= r^{10} - 2kr^5x^5 + x^{10}$ (the Value of n, to render the Operation more perſpicuous, being firſt expreſſed by 5.)

Then, by Multiplication and Tranſpoſition, we ſhall have

$$\left.\begin{array}{l} r^{10}+Ar^9x+Br^8x^2+Cr^7x^3+Dr^6x^4+Cr^5x^5+Br^4x^6+Ar^3x^7+r^2x^8\ldots * \ldots\ldots * \\ * -2ar^9x-2aAr^8x^2-2aBr^7x^3-2aCr^6x^4-2aDr^5x^5-2aCr^4x^6-2aBr^3x^7-2aAr^2x^8-2arx^9\ldots * \\ * \quad * \quad +r^8x^2+Ar^7x^3+Br^6x^4+Cr^5x^5+Dr^4x^6+Cr^3x^7+Br^2x^8+Arx^9+x^{10} \\ -r^{10}\ldots\ldots\ldots\ldots\ldots\ldots+2kr^5x^5\ldots\ldots\ldots\ldots\ldots\ldots-x^{10} \end{array}\right\} = 0$$

Whence, $A=2a$, $B=2Aa-1$, $C=2aB-A$, $D=2aC-B$, and $2C-2aD+k=0$. But, if Q be taken to denote the Arch (EF) of a Circle EHK, whoſe Radius EO is Unity, and Co-ſine $(Of)=a$; and s be put for (Ff) the Sine of the ſame Arch; then (*by Corol. to Lem.* 1.) $sA=\text{Sin. }2Q$, $sB=\text{Sin. }3Q$, $sC=\text{Sin. }4Q$ &c. and conſequently $A=\frac{\text{Sin. }2Q}{s}$, $B=\frac{\text{Sin. }3Q}{s}$, $C=\frac{\text{Sin. }4Q}{s}$, $D=\frac{\text{Sin. }5Q}{s}$ (or, $\frac{\text{Sin. }nQ}{s}$). Moreover, becauſe, $2C-2aD+2k=0$, or $D\times a-C\times 1=k$, where (as appears from above) $D\times a-C\times 1=\frac{\text{Sin. }5Q\times\text{Co-ſ. }Q-\text{Sin. }4Q\times\text{Rad.}}{s}=\text{Co-ſ. }5Q$ (*by Caſe* 1. *Lem.* 2.) we therefore have $\text{Co-ſ. }5Q\ (nQ)=k$. Whence this Conſtruction.



Take R to denote the Arch (EM) whoſe Co-ſine (ON) is the given Co-efficient k, and let Q (EF) be taken to EM as 1 to n; then the Co-ſine (Of) of this laſt Arch will be the true Value of a. But this is only one of the Values that a will admit of: For it is well known, that the Co-ſine of any Arch, is alſo the Co-ſine of the ſame Arch increaſed by any Number of Times the whole Periphery (P). Therefore, ſeeing the Co-ſine of nQ (= Co-ſine of R) is likewiſe = Co-ſine $\overline{P+R}$ = Co-ſ. $\overline{2P+R}$ = Coſ. $\overline{3P+R}$ &c. it follows that Q (whoſe Co-ſine is a) will be expreſſed by any one of the Arcs, $\frac{R}{n}$, $\frac{P+R}{n}$, $\frac{2P+R}{n}$, $\frac{3P+R}{n}$ &c. (or by EG, EH, EI,

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&c. ſuppoſing the whole Periphery to be divided into n equal Parts, from the Point F). Hence, if the Co-ſines of theſe ſeveral Arcs, expreſſing all the different Values of a, be repreſented by b, c and d, *&c.* reſpectively, we ſhall have $r^2-2brx+x^2$, $r^2-2crx+x^2$, $r^2-2drx+x^2$, *&c.* for the ſeveral required Factors, by which $r^{2n}-2kr^n x^n+x^{2n}$ may be reſolved: and conſequently $\overline{r^2-2brx+x^2}\times\overline{r^2-2crx+x^2}\times\overline{r^2-2drx+x^2}\ (n) = r^{2n}-2kr^n x^n+x^{2n}$. *Q. E. I.*

Note, If the Sign of the middle Term $2kr^n x^n$ be poſitive, the Diſtance (or Co-ſine) ON muſt be taken on the contrary Side of the Center: But when k is greater than Unity, this Method of Solution fails; ſince no Co-ſine can be greater than the Radius.

Corollary I.

315. If $k=1$, the Arch R (whoſe Co-ſine is k) being $=0$, the Values of b, c, d, *&c.* will be expreſſed by the Co-ſines of the Arcs $\frac{0}{n}$, $\frac{P}{n}$, $\frac{2P}{n}$, $\frac{3P}{n}$ *&c.* reſpectively: And our general Equation will here become $r^{2n}-2r^n x^n+x^{2n}=\overline{r^2-2brx+x^2}\times\overline{r^2-2crx+x^2}\ \overline{r^2-2drx+x^2}\ (n)$. From whence, by extracting the Square-Root, on both Sides, we alſo have $r^n \backsim x^n = \overline{r^2-2brx+x^2}\big)^{\frac{1}{2}}\times\overline{r^2-2crx+x^2}\big)^{\frac{1}{2}}\ (n)$.

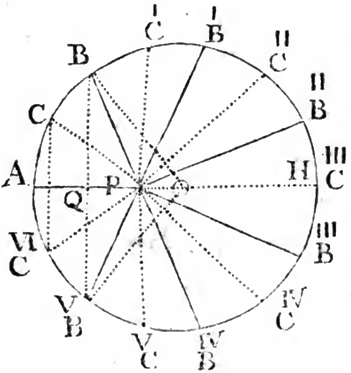
Corollary II.

316. But, if $k=-1$ (or the middle Term be $+2r^n x^n$) then the Arch R being $=\frac{P}{2}$, the Values of b, c, d, *&c.* will, *here*, be defined by the Co-ſines of

the Arcs $\frac{P}{2n}$, $\frac{3P}{2n}$, $\frac{5P}{2n}$, *&c.* and our Equation, by taking the Root, as above, will become $r^n + x^n = \overline{r^2 - 2brx + x^2}\big|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}\big|^{\frac{1}{2}}$ (n).

SCHOLIUM.

317. From the two preceding Corollaries, the Demonſtration of that remarkable Property of the Circle given, and apply'd to finding a vaſt Number of Fluents, in *Cotes's Harmonia Menſurarum*, is very eaſily, and naturally, deduced.



For, let the Periphery of the Circle ABB' *&c.* whoſe Radius is expreſſed by r, be divided into as many equal Parts AB, BB', $B'B''$, *&c.* as there are Units in the given Integer n; ſo that AB, AB', AB'', *&c.* may reſpectively exhibit the Values of the foreſaid Arcs $\frac{P}{n}$, $\frac{2P}{n}$, $\frac{3P}{n}$ *&c.* (*vid. Corol.* 1.) Moreover, let OQ be the Co-ſine of the firſt of them; and, in the Radius OA (produced if neceſſary) let there be taken $OP = x$; and let OB, QB, PB, *&c.* *&c.* be drawn: Then, the Co-ſine of the Angle AOB $(= \frac{P}{n})$ to the Radius 1, being expreſſed by c (*vid. Corol.* 1.) it will be $1 : c :: r\ (OB) : OQ = cr$: Whence PB^2 ($= OB^2 + OP^2 - 2OQ \times OP$) $= r^2 + x^2 - 2crx = r^2 - 2crx + x^2$.

By the very ſame Argument $P\overset{\prime}{B}{}^2$ is $= r^2 - 2drx + x^2$, &c. &c. Therefore, becauſe $r^n \backsim x^n = \overline{r^2 - 2brx + x^2}^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}^{\frac{1}{2}} \times \overline{r^2 - 2drx + x^2}^{\frac{1}{2}}$ (*n*), *by Corol.* 1. it follows that their Equals, $AO^n \backsim OP^n$ and $PA \times PB \times P\overset{\prime}{B} \times P\overset{\prime\prime}{B}$ &c. muſt be equal likewiſe: *Which is the firſt Part of the Theorem above hinted at.*

After the ſame Manner, if the Arcs AC, $A\overset{\prime}{C}$, $A\overset{\prime\prime}{C}$, $A\overset{\prime\prime\prime}{C}$ be taken reſpectively equal to $\frac{P}{2n}$, $\frac{3P}{2n}$, $\frac{5P}{2n}$ &c. it will appear (*from Corol.* 2.) that $AO^n + PO^n$ is $= PC \times P\overset{\prime}{C} \times P\overset{\prime\prime}{C}$ (*n*) *Which is the latter Part of the ſame Theorem.*

Hence (by the Bye) all the Roots of the Equation $x^n = r^n$ are very readily found: For, ſince $AO^n \backsim PO^n = PA \times PB \times P\overset{\prime}{B}$ &c. where the ſecond Factor and the laſt, the third and the laſt but one, &c. are reſpectively equal to each other, it is evident that $AO^n \backsim PO^n$ ($r^n \backsim x^n$) is alſo $= PA \times PB^2 \times P\overset{\prime}{B}{}^2 \times P\overset{\prime\prime}{B}{}^2 = \overline{r \backsim x} \times \overline{r^2 - 2crx + x^2} \times \overline{r^2 - 2drx + x^2}$ &c. Whence, $x^n \backsim r^n$ being $= 0$, it follows that $\overline{r \backsim x} \times \overline{r^2 - 2crx + x^2}$ &c. is $= 0$: From which, by extracting the Roots out of the Equations $r \backsim x = 0$, $r^2 - 2crx + x^2 = 0$, $r^2 - 2drx + x^2 = 0$ &c. we get r, $r \times \overline{c + \sqrt{c^2 - 1}}$, $r \times \overline{c - \sqrt{c^2 - 1}}$, $r \times \overline{d + \sqrt{d^2 - 1}}$, &c.

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&c. for the ſeveral Roots of the Equation $x^n = r^n$; whereof the firſt, only, is poſſible.

By the ſame Way of proceeding all the Roots of the Equation, $x^n + r^n = 0$, will alſo be found: For, ſeeing $x^n + r^n$ is $= \overline{r^2 - 2brx + x^2}\big)^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}\big)^{\frac{1}{2}}$ *&c.* $(= \mathrm{PC} \times \mathrm{P\overset{\prime}{C}} \times \mathrm{P\overset{\prime\prime}{C}}$ *&c.*) where the firſt Factor and the laſt, the ſecond and the laſt but one, *&c.* are reſpectively equal to each other, it is plain that $x^n + r^n$ is likewiſe $= \overline{r^2 - 2brx + x^2} \times \overline{r^2 - 2crx + x^2}$ *&c.* and conſequently $x = r \times \overline{b \pm \sqrt{b^2 - 1}}$ *&c.&c.* Where the Roots are all impoſſible; except the laſt, when their Number (n) is odd.

LEMMA IV.

318. *Suppoſing every thing to remain as in the preceding* Lemma, *and that* $\dot{k}$, $\dot{b}$, $\dot{c}$, $\dot{d}$ &c. *denote the Sines of the Arcs* R, $\frac{R}{n}$, $\frac{P+R}{n}$, $\frac{2P+R}{n}$ *&c.* (*whoſe Co-ſines are* k, b, c, d, &c.) *then, I ſay, the Fraction* $\frac{n\dot{k}r^n x^n}{r^{2n} - 2kr^n x^n + x^{2n}}$ *is equal to* $\frac{\dot{b}rx}{r^2 - 2brx + x^2} + \frac{\dot{c}rx}{r^2 - 2crx + x^2} + \frac{\dot{d}rx}{r^2 - 2drx + x^2}$ *&c.*

For, ſince $r^{10} - 2kr^5x^5 + x^{10}$ $(r^{2n} - 2kr^n x^n + x^{2n})$ is $= \overline{r^2 - 2arx + x^2} \times \overline{r^8 + Ar^7x + Br^6x^2 + Cr^5x^3 + Dr^4x^4 + Cr^3x^5 + Br^2x^6 + Arx^7 + x^8}$ (*by the foreſaid* Lemma) and it is alſo proved that $A = \frac{\text{Sin. } 2Q}{s}$, $B = \frac{\text{Sin. } 3Q}{s}$, $C =$

$C = \frac{\text{Sin. } 4Q}{s}$ &c. it is evident, therefore, that $\frac{r^{10}-2kr^5x^5+x^{10}}{r^2-2arx+x^2}$ $(=r^8+Ar^7x+Br^6x^2$ &c.) is $=r^8+\frac{\text{Sin. } 2Q}{s}\times r^7x$ &c. and conſequently $\frac{s\times\overline{r^{10}-2kr^5x^5+x^{10}}}{r^2-2arx+x^2}$ $=\text{Sin. } Q\times r^8+\text{Sin. } 2Q\times r^7x+\text{Sin. } 3Q\times r^6x^2+\text{Sin. } 4Q\times r^5x^3+\text{Sin. } 5Q\times r^4x^4+\text{Sin. } 4Q\times r^3x^5$ &c. In which Equation, for a and s, let their ſeveral reſpective Values b, c, d, &c. and $\acute{b}$, $\acute{c}$, $\acute{d}$, &c. be, ſucceſſively, ſubſtituted; and let the correſponding Arcs $\frac{R}{n}$, $\frac{P+R}{n}$, $\frac{2P+R}{n}$ &c. be repreſented by Q, Q', Q'', &c. then we ſhall have

$$\frac{\acute{b}\times\overline{r^{10}-2kr^5x^5+x^{10}}}{r^2-2brx+x^2}=\text{Sin. } Q\times r^8+\text{Sin. } 2Q\times r^7x \text{ \&c.}$$

$$\frac{\acute{c}\times\overline{r^{10}-2kr^5x^5+x^{10}}}{r^2-2crx+x^2}=\text{Sin. } Q'\times r^8+\text{Sin. } 2Q'\times r^7x \text{ \&c.}$$

&c. &c.

Which Equations, added all together, give

$$\overline{r^{10}-2kr^5x^5+x^{10}}\times\overline{\frac{\acute{b}}{r^2-2brx+x^2}+\frac{\acute{c}}{r^2-2crx+x^2}+\frac{\acute{d}}{r^2-2drx+x^2}\text{ \&c.}}$$

$$= \left\{\begin{matrix} \text{Sin. } \mathcal{Q} \\ \text{Sin. } \mathcal{Q}' \\ \text{Sin. } \mathcal{Q}'' \\ \text{Sin. } \mathcal{Q}''' \\ \text{Sin. } \mathcal{Q}'''' \end{matrix}\right\} \times r^8 + \left\{\begin{matrix} \text{Sin. } 2\mathcal{Q} \\ \text{Sin. } 2\mathcal{Q}' \\ \text{Sin. } 2\mathcal{Q}'' \\ \text{Sin. } 2\mathcal{Q}''' \\ \text{Sin. } 2\mathcal{Q}'''' \end{matrix}\right\} \times r^7 x + \ldots\ldots + \left\{\begin{matrix} \text{Sin. } n\mathcal{Q} \\ \text{Sin. } n\mathcal{Q}' \\ \text{Sin. } n\mathcal{Q}'' \\ \text{Sin. } n\mathcal{Q}''' \\ \text{Sin. } n\mathcal{Q}'''' \end{matrix}\right\} \times r^4 x^4 + \text{\&c.}$$

But the Sines of the firſt Column, being thoſe of an arithmetical Progreſſion (whoſe common Difference is $\frac{P}{n}$) by which the whole Periphery is divided into n (5) equal Parts, their Sum will therefore, it is well known, be equal to Nothing; or all the negative ones equal to all the poſitive ones.

The ſame is alſo true with regard to the Sines of the ſecond Column; whoſe Arcs $\frac{2R}{n}$, $\frac{2P+2R}{n}$, $\frac{4P+2R}{n}$ *&c.* (having $\frac{2P}{n}$ for their common Difference) divide the Periphery (twice taken) into the ſame Number (n) of equal Parts. But the Sines of the middle Column (which is the laſt above exhibited) will not vaniſh, as all the reſt do: For, nQ being $=R$, $n\acute{Q}=P+R$, $n\ddot{Q}=2P+R$, *&c.* the common Difference will here be equal to (P) the whole Periphery; and therefore, every Arch terminating in the ſame Point with the firſt, the Circle will, in this Caſe, remain undivided, and the Sine of each be equal to ($\acute{k}$) the Sine of the firſt.

Hence, our Equation is reduced to $\overline{r^{10}-2kr^5x^5+x^{10}}\times$

$$\overline{\frac{\acute{b}}{r^2-2brx+x^2}+\frac{\acute{c}}{r^2-2crx+x^2}} \ \&c. = 5\acute{k}r^4x^4;$$ which divided by $r^{10}-2kr^5x^5+x^{10}$, and multiply'd by rx, gives

$$\frac{\acute{b}rx}{r^2-2brx+x^2}+\frac{\acute{c}rx}{r^2-2crx+x^2}+\frac{\acute{d}rx}{r^2-2drx+x^2}\ \&c. =$$

$$\frac{5\acute{k}r^5x^5}{r^{10}-2kr^5x^5+x^{10}}=\frac{n\acute{k}r^nx^n}{r^{2n}-2kr^nx^n+x^{2n}}. \quad Q.E.D.$$

The ſame otherwiſe.

319. Since $r^{2n}-2kr^nx^n+x^{2n}$ is $=\overline{r^2-2brx+x^2}\times\overline{r^2-2crx+x^2}\times\overline{r^2-2drx+x^2}$ (n) *by Lemma* 3. it is evident that, Log. $\overline{r^{2n}-2kr^nx^n+x^{2n}}=$ Log. $\overline{r^2-2brx+x^2}+$Log.$\overline{r^2-2crx+x^2}+$Log.$\overline{r^2-2drx+x^2}$ (n). And, as this Equation holds univerſally, let k and x be what they will (which two Quantities may be ſuppoſed to flow independently of each other) let the

Fluxion of the *whole* Equation be taken, making k variable (and x conſtant); which gives $\frac{-2\dot{k}r^n x^n}{r^{2n}-2kr^n x^n+x^{2n}}$

$$= \frac{-2\dot{b}rx}{r^2-2brx+x^2} - \frac{2\dot{c}rx}{r^2-2crx+x^2} - \frac{2\dot{d}rx}{r^2-2drx+x^2}$$

(n)*. But, k, b, c, d, &c. are the Coſines of the Arcs R, $\frac{R}{n}$, $\frac{R+P}{n}$, $\frac{R+2P}{n}$ &c. (whereof the correſponding Sines are $\acute{k}$, $\acute{b}$, $\acute{c}$, &c.) therefore, the Fluxion of the firſt of theſe Arcs being denoted by $\dot{R}$, the Fluxion of each of the reſt will be expreſſed by $\frac{\dot{R}}{n}$: And ſo (the Fluxion of the Co-ſine of an Arch being equal to the Fluxion of the Arch itſelf drawn into its Sine, apply'd to Radius*) it follows that $\dot{k} = \dot{R}\acute{k}$, $\dot{b} = \frac{\dot{R}}{n} \times \acute{b}$, $\dot{c} = \frac{\dot{R}}{n} \times \acute{c}$, &c. Which Values being ſubſtituted in the foregoing Equation, and the whole divided by $\frac{-2\dot{R}}{n}$, we have $\frac{n\acute{k}r^n x^n}{r^{2n}-2kr^n x^n+x^{2n}} = \frac{\acute{b}rx}{r^2-2brx+x^2} + \frac{\acute{c}rx}{r^2-2crx+x^2} + \frac{\acute{d}rx}{r^2-2drx+x^2}$ (n).

LEMMA V.

320. *To determine the Series, ariſing from the Diviſion of Unity by a Trinomial, $x^2-2arx+r^2$; and to exhibit the Remainder after any given Number (v) of Terms in the Quotient.*

Let $x^{-2}+Arx^{-3}+Br^2x^{-4}+Cr^3x^{-5}+Dr^4x^{-6}$ repreſent the required Quotient continued to 5 Terms $(v,$

* Art. 126.

* Art. 142.

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(v, to render the Proceſs the more obvious, being firſt expounded by that Number) and let $Er^5x^{-5}+Fr^6x^{-6}$ be the Remainder. Then, becauſe $\frac{1}{x^2-2arx+r^2}$ is $= x^{-2}+Arx^{-3}+Br^2x^{-4}+Cr^3x^{-5}+Dr^4x^{-6}+\frac{Er^5x^{-5}+Fr^6x^{-6}}{x^2-2arx+r^2}$, we ſhall, by reducing the whole Equation to one Denomination, have

$$\left.\begin{array}{llllllll} & 1 & +Arx^{-1} & +Br^2x^{-2} & +Cr^3x^{-3} & +Dr^4x^{-4} & +Er^5x^{-5} & +Fr^6x^{-6} \\ * & -2arx^{-1} & -2aAr^2x^{-2} & -2aBr^3x^{-3} & -2aCr^4x^{-4} & -2aDr^5x^{-5} & * & \\ 1 & * & +r^2x^{-2} & +Ar^3x^{-3} & +Br^4x^{-4} & +Cr^5x^{-5} & +Dr^6x^{-6} & \end{array}\right\} = 0$$

Whence $A=2a$, $B=2aA-1$, $C=2aB-A$, $D=2aC-B$, $E=2aD-C$, and $F=-D$.

Therefore, if Q be now put for the Arch whoſe Radius is 1 and Co-ſine a, and there be taken $S = Sin.\ Q$, $\overset{\prime}{S} = Sin.\ 2Q$, $\overset{\prime\prime}{S} = Sin.\ 3Q$, *&c.* we ſhall, alſo, have $A\ (2a) = \frac{\overset{\prime}{S}}{S}$, $B = \frac{\overset{\prime\prime}{S}}{S}$, $C = \frac{\overset{\prime\prime\prime}{S}}{S}$, $D = \frac{\overset{\prime\prime\prime\prime}{S}}{S}$, $E = \frac{\overset{v}{S}}{S} = \frac{Sin.\ 6Q}{S}$, $F\ (-D) = -\frac{Sin.\ 5Q}{S}$ (*by Corol. to Lem.* 1.) And conſequently $\frac{1}{x^2 - 2arx + r^2} =$

$$\frac{Sx^{-2} + \overset{\prime}{S}rx^{-3} + \overset{\prime\prime}{S}r^2x^{-4} + \overset{\prime\prime\prime}{S}r^3x^{-5} + \overset{\prime\prime\prime\prime}{S}r^4x^{-6}}{S} + \frac{Sin.\ 6Q \times r^5x^{-5} - Sin.\ 5Q \times r^6x^{-6}}{S \times \overline{x^2 - 2arx + r^2}}.$$

Whence, *univerſally*, $\frac{1}{x^2 - 2arx + r^2} =$

$$\frac{Sx^{-2} + \overset{\prime}{S}rx^{-3} + \overset{\prime\prime}{S}r^2x^{-4} + \overset{\prime\prime\prime}{S}r^3x^{-5}\ \text{\&c. (to } v \text{ Terms)}}{S} + \frac{Sin.\ \overline{v+1}.\, Q \times r^v x^{-v} - Sin.\ vQ \times r^{v+1} x^{-v-1}}{S \times \overline{x^2 - 2arx + r^2}}.$$

Which laſt Equation (tho' obvious enough from the preceding one) may be inveſtigated in a general Manner (if required) by aſſuming $x^{-2} + Arx^{-3} + Br^2x^{-4} + Cr^3x^{-5} + \ldots\ldots dr^{v-2}x^{-v} + er^{v-1}x^{-v-1} + \frac{fr^v x^{-v} + gr^{v+1}x^{-v-1}}{x^2 - 2arx + r^2} = \frac{1}{x^2 - 2arx + r^2}$, and proceeding as above: By which Means you will find $A = 2a$, $B = 2aA - 1$, *&c.* $f = 2ae - d = \frac{Sin.\ \overline{v+1} \times Q}{S}$; and $g\ (= -e) = -\frac{Sin.\ vQ}{S}$. And thus may the third

Lemma be made out, if any Objection, or Difficulty, ſhould ariſe about its being general.

COROLLARY.

321. If, in the given Trinomial $x^2-2arx+r^2$, we ſuppoſe r^2, inſtead of x^2, to be the leading Term whereby the Quotient is produced; then, ſince r and x are affected exactly alike; we ſhall, by writing r for x, and x for r, have $\frac{1}{r^2-2axr+x^2}=$

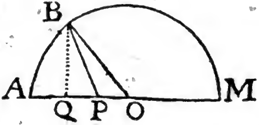
$$\frac{Sr^{-2}+\acute{S}xr^{-3}+\acute{\acute{S}}x^2r^{-4}\quad(v)}{S}+$$

$$\frac{\overline{Sin.\ \overline{v+1}\times\mathcal{Q}}\times x^{v}r^{-v}-\overline{Sin.\ v\mathcal{Q}}\times x^{v+1}\times r^{-v-1}}{S\times\overline{r^2-2axr+x^2}}.$$

PROB. I.

322. *To find the Fluent of* $\frac{\dot{x}}{rr-2arx+xx}$, *together with that of* $\frac{x\dot{x}}{rr-2arx+xx}$.

Let ABM *&c.* be a Circle whoſe Radius OA (or OM) is r, and let the Angle AOB be ſuch, that its Co-ſine, to the Radius 1, may be equal to a; or ſo, that OQ (ſuppoſing BQ perpendicular to OA) may be $=ar$: Moreover let s denote the Sine of the ſaid Angle AOB, correſponding to the Co-ſine a, and let OP (conſidered as variable by the Motion of P along OA) expreſs the Value of x: Then, PB² (OB²+OP²−2OQ×OP) $=rr-2arx+xx$: And the Fluxion of the Meaſure of the Angle QBP (Radius being Unity) will be repreſented



ſented by $\frac{BQ \times Flux.\ QP}{BP^2}$ (*vid. Art.* 142.) or by $\frac{rs \times -\dot{x}}{rr-2arx+xx}$; and conſequently *that* of OBP, by $\frac{rs\dot{x}}{rr-2arx+xx}$: Whence it is evident that the Fluent of $\frac{\dot{x}}{rr-2arx+xx}$ (contemporaneous with x) is truly expreſſed by $\frac{1}{rs} \times OBP$.

Again, ſince $\frac{x\dot{x}}{rr-2arx+xx}$ may be transformed to $\frac{-ar\dot{x}+x\dot{x}}{rr-2arx+xx} + \frac{ar\dot{x}}{rr-2arx+xx}$; where the Fluent of the former Part is $= \frac{1}{2}$ hyp. Log. $\frac{rr-2arx+xx}{rr}$ * $= \frac{1}{2}\,hyp.\,Log.\,\frac{PB^2}{OB^2} = hyp.\ Log.\ \frac{PB}{OB}$; and that of the latter Part $= \frac{a}{s} \times OBP$; it appears that the Fluent of $\frac{x\dot{x}}{rr-2arx+xx}$ is truly expounded by *hyp. Log.* $\frac{PB}{OB}$, $+ \frac{a}{s} \times OBP$. *Q. E. I.*

* Art. 126.

COROLLARY

323. Since, $PB : PO :: Sin.\ BOP\ (s) : Sin.\ OBP = \frac{sx}{\sqrt{rr-2arx+xx}}$; it follows, if the hyperbolical Logarithm of $\frac{\sqrt{r^2-2arx+xx}}{r}$, be repreſented by M, and the Arch, whoſe Sine is $\frac{sx}{\sqrt{rr-2arx+xx}}$ and Radius Unity,

Unity by N, that the Fluents of $\frac{\dot{x}}{rr-2arx+xx}$ and $\frac{x\dot{x}}{rr-2arx+xx}$ will be expreſſed by $\frac{N}{rs}$ and $M+\frac{aN}{s}$ reſpectively.

PROB. II.

324. *To determine the Fluent of* $\frac{x^m\dot{x}}{x^2-2arx+r^2}$; *ſuppoſing* m *any whole poſitive Number, and* a *leſs than Unity.*

Let every thing remain as in *Lemma* 5. and then, if the Equation there brought out be multiply'd by $x^m\dot{x}$, and v at the ſame time be expounded by $m-1$, we ſhall get

$$\frac{x^m\dot{x}}{x^2-2arx+r^2}=\frac{Sx^{m-2}\dot{x}+\overset{\prime}{S}rx^{m-3}\dot{x}+\overset{\prime\prime}{S}r^2x^{m-4}\dot{x}}{S}$$

$$(m-1)+\frac{\overline{\text{Sin. }mQ}\times r^{m-1}x\dot{x}-\overline{\text{Sin. }\overline{m-1}\times Q}\times r^m\dot{x}}{S\times\overline{xx-2arx+rr}}$$

Whoſe Fluent will therefore be given by the preceding Propoſition: For, ſuppoſing the Values of M and N to be as there ſpecified, the Fluent of the laſt Term $\left(\frac{\overline{\text{Sin. }mQ}\times r^{m-1}x\dot{x}-\overline{\text{Sin. }\overline{m-1}\times Q}\times r^m\dot{x}}{S\times\overline{xx-2arx+rr}}\right)$ will, it is manifeſt*, be expreſſed by $\frac{1}{S}$ *into* $\overline{\text{Sin. }mQ}\times r^{m-1}\times$

$$\overline{M+\frac{aN}{S}}-\overline{\text{Sin. }\overline{m-1}\times Q}\times r^m\times\frac{N}{rS}=\frac{r^{m-1}}{S}\text{ into}$$

$$\overline{\text{Sin. }mQ}\times M+\frac{\overline{\text{Sin. }mQ\times a}-\overline{\text{Sin. }\overline{m-1}\times Q}}{S}\times N$$

$$=\frac{r^{m-1}}{S}\text{ into }\overline{\text{Sin. }mQ}\times M+\overline{\text{Co-ſ. }mQ}\times N\ (\textit{by Lem.2.}$$

Caſe

* Art. 323.

Caſe 1.) To which adding the Fluent of the preceding Series, there reſults $\frac{1}{S}\times\overline{\frac{Sx^{m-1}}{m-1}+\frac{\overset{\prime}{S}rx^{m-2}}{m-2}+\frac{\overset{\prime\prime}{S}r^{2}x^{m-3}}{m-3}\ (m-1)}$ $+\frac{r^{m-1}}{S}\times\overline{\overline{\text{Sin. }m\mathcal{Q}}\times M+\overline{\text{Co-ſ. }m\mathcal{Q}}\times N}$. $\mathcal{Q}$. *E. I.*

COROLLARY.

325. Hence, the Fluent of $\frac{-ax^{m}\dot{x}+rx^{m-1}\dot{x}}{xx-2arx+rr}$ may be deduced: For, by writing $m-1$, inſtead of m, the Fluent of $\frac{x^{m-1}\dot{x}}{xx-2arx+rr}$ will be found $=\frac{1}{S}\times\overline{\frac{Sx^{m-2}}{m-2}+\frac{\overset{\prime}{S}rx^{m-3}}{m-3}+\frac{\overset{\prime\prime}{S}r^{2}x^{m-4}}{m-4}\ (m-2)}+\frac{r^{m-2}}{S}\times$ $\overline{\overline{\text{Sin. }\overline{m-1}\times\mathcal{Q}}\times M+\overline{\text{Co-ſ. }\overline{m-1}\times\mathcal{Q}}\times N}$: Which Fluent being multiply'd by r, and *that* of $\frac{x^{m}\dot{x}}{xx-2arx+rr}$ (given above) by $-a$, we ſhall, when the homologous Terms are united, have $\frac{1}{S}\times\overline{-aS\times\frac{x^{m-1}}{m-1}-\overline{a\overset{\prime}{S}-S}\times\frac{rx^{m-2}}{m-2}-\overline{a\overset{\prime\prime}{S}-\overset{\prime}{S}}\times\frac{r^{2}x^{m-3}}{m-3}\ (m-1)}+\frac{r^{m-1}}{S}$ into $-$ $\overline{\overline{\text{Sin. }m\mathcal{Q}}\times a-\text{Sin. }\overline{m-1}\times\mathcal{Q}}\times M-\overline{\overline{\text{Co-ſ.}m\mathcal{Q}}\times a-\text{Co-ſ. }\overline{m-1}\times\mathcal{Q}}\times N$, for the true Fluent of the Quantity propounded.

But (*by Caſe* 1. *Lem.* 2.) $\frac{a\overset{\prime}{S}-S}{S}$ $\left(=\frac{\text{Sin. }2\mathcal{Q}\times a-\text{Sin }\mathcal{Q}\times\text{Rad.}}{\text{Sin. }\mathcal{Q}}\right)=\text{Co-ſ. }2\mathcal{Q}$: Alſo

$\frac{a\overset{\prime\prime}{S}-\overset{\prime}{S}}{S}\ \left(\frac{\text{Sin. }3Q\times a-\text{Sin. }2Q\times\text{Rad.}}{S}\right)=\text{Co-ſ. }3Q$

&c. And, by *Caſe* 2. of the ſame *Lemma*,

$\frac{\text{Co-ſ. }\overline{m-1}\times Q-\overline{\text{Coſ.}mQ}\times a}{S}=\text{Sin. }mQ$: Whence, by ſubſtituting theſe Values, our Fluent is reduced to

$$-\text{Coſ. }Q\times\frac{x^{m-1}}{m-1}-\text{Coſ. }2Q\times\frac{rx^{m-2}}{m-2}-\text{Co-ſ. }3Q\times\frac{r^2x^{m-3}}{m-3}-\text{Co-ſ. }4Q\times\frac{r^3x^{m-4}}{m-4}\ (m-1)-r^{m-1}\times\overline{\overline{\text{Co-ſ. }mQ}\times M-\overline{\text{Sin. }MQ}\times N}.$$

PROB. III.

326. *To determine the Fluent of* $\frac{x^{-m}\dot{x}}{r^2-2arx+x^2}$; *under the Reſtrictions ſpecified in the preceding Problem.*

If the Equation in *Art.* 321. be multiply'd by $x^{-m}\dot{x}$, and v at the ſame time be expounded by m, we ſhall have $\frac{x^{-m}\dot{x}}{r^2-2arx+x^2}=$

$$\frac{Sr^{-2}x^{-m}\dot{x}+\overset{\prime}{S}r^{-3}x^{1-m}\dot{x}+\overset{\prime\prime}{S}r^{-4}x^{2-m}\dot{x}}{S}\ (m)+\frac{r^{-m-1}}{S}\times\frac{\overline{\text{Sin. }\overline{m+1}\times Q}\times r\dot{x}-\overline{\text{Sin. }mQ}\times x\dot{x}}{r^2-2arx+x^2}:$$

Where, the Fluent of the laſt Term being $\frac{r^{-m-1}}{S}\times\overline{\text{Sin. }\overline{m+1}\times Q\times\frac{N}{S}-\overline{\text{Sin. }mQ}\times\overline{M+\frac{aN}{S}}}$ * $=$ * Art. 323.

$\frac{r^{-m-1}}{S}$ into $-\ \overline{\text{Sin. }mQ}\times M+$

$$\frac{\text{Sin. }\overline{m+1\times\mathcal{Q}}-\text{Sin. }m\mathcal{Q}\times a}{S}\times N=\frac{r^{-m-1}}{S}\times$$

$$\overline{-\overline{\text{Sin. }m\mathcal{Q}}\times M+\overline{\text{Co-ſ. }m\mathcal{Q}}\times N}$$ (*by Caſe* 3. *Lem.* 2.) it follows that the Fluent of the whole Expreſſion, or the Quantity ſought, will be truly expreſſed by

$$\frac{1}{S}\times\frac{Sr^{-2}x^{1-m}}{1-m}+\frac{\overset{\prime}{S}r^{-3}x^{2-m}}{2-m}+ \text{\&c. or its Equal}$$

$$\frac{-1}{S}\times\frac{Sx^{1-m}}{\overline{m-1}.r^{2}}+\frac{\overset{\prime}{S}x^{2-m}}{\overline{m-2}.r^{3}}+\frac{\overset{\prime\prime}{S}x^{3-m}}{\overline{m-3}.r^{4}}\ (m)+$$

$$\frac{1}{Sr^{m+1}}\times\overline{\overline{\text{Co-ſin. }m\mathcal{Q}}\times N-\overline{\text{Sin. }m\mathcal{Q}}\times M}.$$

PROB. IV.

327. *To find the Fluent of* $\frac{x^{m-1}\dot{x}}{r^{n}+x^{n}}$; *m and n being any whole poſitive Numbers, whereof the former does not exceed the latter.*

Let b, c, d, *&c.* denote the Co-ſines of the Arcs $\frac{360°}{2n}$, $\frac{3\times 360°}{2n}$, $\frac{5\times 360°}{2n}$, *&c.* (Radius being Unity) Then (*by Corol.* 2. *Lem.* 3.) we ſhall have $r^{n}+x^{n}=$ $\overline{rr-2brx+xx}|^{\frac{1}{2}}\times\overline{rr-2crx+xx}|^{\frac{1}{2}}\times\overline{rr-2drx+xx}|^{\frac{1}{2}}$ (n). Whence $Log.\ \overline{r^{n}+x^{n}}=\frac{1}{2}Log.\ \overline{rr-2brx+xx}+\frac{1}{2}Log.\ \overline{rr-2crx+xx}+\frac{1}{2}Log.\ \overline{rr-2drx+xx}\ (n)$ and, conſequently, by taking the Fluxion, on both Sides, $\frac{nx^{n-1}\dot{x}}{r^{n}+x^{n}}=\frac{x\dot{x}-br\dot{x}}{xx-2brx+rr}+\frac{x\dot{x}-cr\dot{x}}{xx-2crx+rr}+\frac{x\dot{x}-dr\dot{x}}{xx-2drx+rr}$ * (n); which laſt Equation, multiply'd by $\frac{x}{\dot{x}}$, gives $\frac{nx^{n}}{r^{n}+x^{n}}=\frac{xx-brx}{xx-2brx+rr}+\frac{xx-crx}{xx-2crx+rr}$

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* Art. 126.

$+\frac{xx-drx}{xx-2drx+rr}$ (n). Let each Side hereof be now ſubtracted from n (or, which comes to the ſame thing, let $\frac{nx^n}{r^n+x^n}$ be taken from n, and each of the (n) Terms on the other Side, from Unity) then we ſhall have $\frac{nr^n}{r^n+x^n}=\frac{-brx+rr}{xx-2brx+rr}+\frac{-crx+rr}{xx-2crx+rr}$ $+\frac{-drx+rr}{xx-2drx+rr}$ (n): Which multiply'd by $\frac{x^{m-1}\dot{x}}{r}$, gives $\frac{nr^{n-1}\times x^{m-1}\dot{x}}{r^n+x^n}=\frac{-bx^m\dot{x}+rx^{m-1}\dot{x}}{xx-2brx+rr}+$ $\frac{-cx^m\dot{x}+rx^{m-1}\dot{x}}{xx-2crx+rr}$ *&c.*

But now, to determine the Fluent hereof, let the ſeveral Arcs $\left(\frac{180^\circ}{n}, \frac{3\times180^\circ}{n}, \frac{5\times180^\circ}{n}\right.$ *&c.*) above ſpecified, be denoted by Q, $\overset{\prime}{Q}$, $\overset{\prime\prime}{Q}$, $\overset{\prime\prime\prime}{Q}$, *&c.* reſpectively; alſo let N, $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$, *&c.* expreſs the Meaſures of the Angles whoſe Sines are $\frac{x\times \text{Sin.}\,Q}{\sqrt{rr-2brx+xx}}$, $\frac{x\times \text{Sin.}\,\overset{\prime}{Q}}{\sqrt{xx-2crx+rr}}$, $\frac{x\times \text{Sin.}\,\overset{\prime\prime}{Q}}{\sqrt{xx-2drx+rr}}$ *&c.* and M, $\overset{\prime}{M}$, $\overset{\prime\prime}{M}$, *&c.* the hyperbolic Logarithms of $\frac{\sqrt{xx-2brx+rr}}{r}$, $\frac{\sqrt{xx-2crx+rr}}{r}$, $\frac{\sqrt{xx-2drx+rr}}{r}$ *&c.* Then (*by Corol. to Prob.* 2.) the Fluent of the firſt Term, $\frac{-bx^m\dot{x}+rx^{m-1}\dot{x}}{xx-2brx+rr}$ (expounding a by b) comes out

$$-\text{Co-ſ.}\ \mathcal{Q}\times\frac{x^{m-1}}{m-1}-\text{Co-ſ.}\ 2\mathcal{Q}\times\frac{rx^{m-2}}{m-2}-\text{Co-ſ.}\ 3\mathcal{Q}\times\frac{r^2x^{m-3}}{m-3}\ (m-1)+r^{m-1}\ \text{into}\ \overline{\text{Sin.}\ m\mathcal{Q}}\times N-\overline{\text{Co-ſ.}\ m\mathcal{Q}}\times M.$$

In the ſame Manner, by writing c for a, $\acute{\mathcal{Q}}$ for $\mathcal{Q}$, $\acute{M}$ for M, and $\acute{N}$ for N) the Fluent of the ſecond Term, $\frac{-cx^m\dot{x}+rx^{m-1}\dot{x}}{xx-2crx+rr}$, is found $=-\text{Co-ſ.}\ \acute{\mathcal{Q}}\times\frac{x^{m-1}}{m-1}-\text{Co-ſ.}\ 2\acute{\mathcal{Q}}\times\frac{rx^{m-2}}{m-2}$ &c. &c.

Therefore the Fluent of the whole Expreſſion, by collecting the homologous Terms, appears to be

$$-\left\{\begin{array}{l}\text{Co-ſ.}\,\mathcal{Q}\\ \text{Co-ſ.}\,\overset{\prime}{\mathcal{Q}}\\ \text{Co-ſ.}\,\overset{\prime\prime}{\mathcal{Q}}\\ \text{Co-ſ.}\,\overset{\prime\prime\prime}{\mathcal{Q}}\\ \text{\&c.}\end{array}\right\}\times\frac{x^{m-1}}{m-1}-\left\{\begin{array}{l}\text{Co-ſ.}\,2\mathcal{Q}\\ \text{Co-ſ.}\,2\overset{\prime}{\mathcal{Q}}\\ \text{Co-ſ.}\,2\overset{\prime\prime}{\mathcal{Q}}\\ \text{Co-ſ.}\,2\overset{\prime\prime\prime}{\mathcal{Q}}\\ \text{\&c.}\end{array}\right\}\times\frac{rx^{m-2}}{m-2}-\left\{\begin{array}{l}\text{Co-ſ.}\,3\mathcal{Q}\\ \text{Co-ſ.}\,3\overset{\prime}{\mathcal{Q}}\\ \text{Co-ſ.}\,3\overset{\prime\prime}{\mathcal{Q}}\\ \text{Co-ſ.}\,3\overset{\prime\prime\prime}{\mathcal{Q}}\\ \text{\&c.}\end{array}\right\}\times\frac{r^2x^{m-3}}{m-3}\ \text{\&c.}$$

$$+\,r^{m-1}\times\left\{\begin{array}{l}\overline{\text{Sin. } m\mathcal{Q}}\times N-\overline{\text{Co-ſ. } m\mathcal{Q}}\times M\\[4pt] \overline{\text{Sin. } m\overset{\prime}{\mathcal{Q}}}\times\overset{\prime}{N}-\overline{\text{Co-ſ. } m\overset{\prime}{\mathcal{Q}}}\times\overset{\prime}{M}\\[4pt] \overline{\text{Sin. } m\overset{\prime\prime}{\mathcal{Q}}}\times\overset{\prime\prime}{N}-\overline{\text{Co-ſ. } m\overset{\prime\prime}{\mathcal{Q}}}\times\overset{\prime\prime}{M}\\[4pt] \overline{\text{Sin. } m\overset{\prime\prime\prime}{\mathcal{Q}}}\times\overset{\prime\prime\prime}{N}-\overline{\text{Co-ſ. } m\overset{\prime\prime\prime}{\mathcal{Q}}}\times\overset{\prime\prime\prime}{M}\\ \qquad\&c. \qquad\qquad\qquad\qquad \&c.\end{array}\right.$$

But the Co-ſines of the firſt Column being thoſe of an arithmetical Progreſſion $\left(\frac{180^\circ}{n}, \frac{3\times 180^\circ}{n}, \frac{5\times 180^\circ}{n}\right.$ &c.) whoſe common Difference is $\frac{360^\circ}{n}$, whereby the whole Periphery is divided into n equal Parts (*vid. Art.* 317.) they will therefore deſtroy one another; ſince it is well known that, if the Periphery of any Circle be divided into any Number (n) of equal Parts, the negative Sines and Co-ſines will be equal to the poſitive ones; which is ſelf-evident when their Number is even.

Hence the Co-ſines in the ſecond and third Columns, &c. will alſo deſtroy one another (*vid. Art.* 318.) But *thoſe* of the laſt Column of all, as well as the Sines, having unequal Multiplicators, muſt remain as above, and that Column, *alone*, (drawn into r^{m-1}) will be the true Fluent of $\frac{nr^{n-1}\times x^{m-1}\dot{x}}{r^n+x^n}$. Whence, putting $m\mathcal{Q}$ $\left(=m\times\frac{180^\circ}{n}\right)=R$, and dividing by nr^{n-1}, we ſhall (becauſe $\overset{\prime}{\mathcal{Q}}=3\mathcal{Q}$, $\overset{\prime\prime}{\mathcal{Q}}=5\mathcal{Q}$, $\overset{\prime\prime\prime}{\mathcal{Q}}=7\mathcal{Q}$ &c.) have

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{l} \overline{\mathit{Sin.}\ R} \times N - \overline{\mathit{Co\text{-}ſ.}\ R} \times M \\ \overline{\mathit{Sin.}\, 3R} \times \overset{\prime}{N} - \overline{\mathit{Co\text{-}ſ.}\, 3R} \times \overset{\prime}{M} \\ \overline{\mathit{Sin.}\, 5R} \times \overset{\prime\prime}{N} - \overline{\mathit{Co\text{-}ſ.}\, 5R} \times \overset{\prime\prime}{M} \\ \overline{\mathit{Sin.}\, 7R} \times \overset{\prime\prime\prime}{N} - \overline{\mathit{Co\text{-}ſ.}\, 7R} \times \overset{\prime\prime\prime}{M} \\ \quad \&c.\ (\text{to } n \text{ Lines.}) \end{array} \right\} = \text{Fluent of } \frac{x^{m-1}\dot{x}}{r^n + x^n}$$

Q. E. I.

COROLLARY.

328 Since the firſt and the laſt, the ſecond and the laſt but one, *&c.* of the foregoing Quantities $x^2 - 2brx + rr$, $xx - 2crx + rr$, $xx - 2drx + rr$ *&c.* are reſpectively equal to each other (*vid. Art.* 317.) the correſponding Fluents, found above, will likewiſe be equal: And therefore the Fluent of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ will, *alſo*, be expreſſed by

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{l} \overline{\mathit{Sin.}\ R} \times 2N - \overline{\mathit{Co\text{-}ſ.}\ R} \times 2M \\ \overline{\mathit{Sin.}\, 3R} \times 2\overset{\prime}{N} - \overline{\mathit{Co\text{-}ſ.}\, 3R} \times 2\overset{\prime}{M} \\ \overline{\mathit{Sin.}\, 5R} \times 2\overset{\prime\prime}{N} - \overline{\mathit{Co\text{-}ſ.}\, 5R} \times 2\overset{\prime\prime}{M} \\ \qquad \&c. \qquad\qquad \&c. \end{array} \right.$$

The Number of Lines to be thus taken being $= \frac{1}{2}n$, when n is even; but, otherwiſe, $= \frac{n+1}{2}$; in which laſt Caſe, the Logarithm, *&c.* in the laſt Line, muſt be taken only once, inſtead of twice; being that of $\frac{r+x}{r}$ (*vid. Art.* 317.)

PROB.

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PROB. V.

329. *To find the Fluent of* $\frac{x^{m-1}\dot{x}}{r^n - x^n}$; m *and* n *being as in the preceding Problem.*

If b, c, d, *&c.* be taken to denote the Co-ſines of the Arcs $\frac{0}{n}$, $\frac{360^\circ}{n}$, $\frac{2\times 360^\circ}{n}$ *&c.* to n Terms, it will appear *(from Corol.* 1. *to Lem.* 3.) that $r^n - x^n$ is $=$ $\overline{rr-2brx+xx}|^{\frac{1}{2}} \times \overline{rr-2crx+xx}|^{\frac{1}{2}} \times \overline{rr-2drx+xx}|^{\frac{1}{2}}$ (n). From whence, by following the Method of the laſt Problem, we alſo have $\frac{nr^{n-1}\times x^{m-1}\dot{x}}{r^n - x^n} =$

$$\frac{-bx^m\dot{x}+rx^{m-1}\dot{x}}{xx-2brx+rr} + \frac{-cx^m\dot{x}+rx^{m-1}\dot{x}}{xx-2crx+rr} \quad \text{\&c.}$$

Which Fluxion having exactly the ſame *Form* with *that* in the preceding Problem, its Fluent will alſo be expreſſed in the very ſame Manner; that is, by

$$r^{m-1} \times \begin{cases} \textit{Sin. } m\mathcal{Q} \times N - \textit{Co-ſ. } m\mathcal{Q} \times M \\ \textit{Sin. } m\overset{\prime}{\mathcal{Q}} \times \overset{\prime}{N} - \textit{Co-ſ. } m\overset{\prime}{\mathcal{Q}} \times \overset{\prime}{M} \\ \textit{Sin. } m\overset{\prime\prime}{\mathcal{Q}} \times \overset{\prime\prime}{N} - \textit{Co-ſ. } m\overset{\prime\prime}{\mathcal{Q}} \times \overset{\prime\prime}{M} \\ \quad (\textit{\&c. to n Lines.}) \end{cases}$$

Only $\mathcal{Q}$, $\overset{\prime}{\mathcal{Q}}$, $\overset{\prime\prime}{\mathcal{Q}}$ *&c.* muſt here ſtand for $\frac{0}{n}$, $\frac{360^\circ}{n}$, $\frac{2\times 360^\circ}{n}$, $\frac{3\times 360^\circ}{n}$ *&c.* (inſtead of $\frac{180^\circ}{n}$, $\frac{3\times 180^\circ}{n}$, $\frac{5\times 180^\circ}{n}$ *&c.*)

There-

Therefore, ſince the multiple Arcs $m\mathcal{Q}$, $m\overset{\prime}{\mathcal{Q}}$, $m\overset{\prime\prime}{\mathcal{Q}}$ *&c.* are, in this Caſe, equal to 0, $m \times \frac{360^\circ}{n}$, $2m \times \frac{360^\circ}{n}$, $3m \times \frac{360^\circ}{n}$ *&c.* (whereof the Sine of the firſt is $=0$, and its Co-ſine $=$ Unity) we ſhall, by putting $R = m \times \frac{360^\circ}{n}$, and dividing the foreſaid Fluent by nr^{n-1}, have

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{l} * \ldots\ldots - \ldots\ldots M \\ Sin.\ R \times \overset{\prime}{N} - Co\text{-}ſ.\ R \times \overset{\prime}{M} \\ Sin.\, 2R \times \overset{\prime\prime}{N} - Co\text{-}ſ.\, 2R \times \overset{\prime\prime}{M} \\ Sin.\, 3R \times \overset{\prime\prime\prime}{N} - Co\text{-}ſ.\, 3R \times \overset{\prime\prime\prime}{M} \\ (\&c.\ to\ n\ Lines.) \end{array} \right\} = \text{Fluent of } \frac{x^{m-1}\dot{x}}{r^n - x^n}$$

Q. E. I.

COROLLARY.

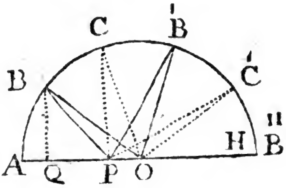
330. Since, in the Fluent here given, the ſecond Line and the laſt, the third and the laſt but one, *&c.* are reſpectively equal (*vid. Art.* 317.) the ſame may alſo be exhibited, thus;

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{l} * \ldots\ldots - \ldots\ldots\quad M \\ Sin.\ R \times 2\overset{\prime}{N} - Co\text{-}ſ.\ R \times 2\overset{\prime}{M} \\ Sin.\ 2R \times 2\overset{\prime\prime}{N} - Co\text{-}ſ.\ 2R \times 2\overset{\prime\prime}{M} \end{array} \right.$$

(*&c.* to $\frac{n+1}{2}$ Lines.)

SCHOLIUM.

331. If the Semi-Periphery ABCH of the Circle whoſe Diameter AH is $2r$, be divided into as many equal



equal Parts AB, BC, $C\acute{B}$, $\acute{B}\acute{C}$ &c. as there are Units in *n* (ſo that $AB = \frac{180^\circ}{n} = \mathcal{Q}$, $A\acute{B} = 3 \times \frac{180^\circ}{n} = \acute{\mathcal{Q}}$ &c. *vid. Art.* 317. and 327.) and in the Radius OA (produced, if neceſſary) there be taken OP$=x$, and PB, OB &c. be drawn, it will appear (from the ſaid Articles, and from *Prop.* 1.) that the Quantities $\sqrt{r^2 - 2brx + xx}$, $\sqrt{r^2 - 2crx + x^2}$ &c. in the former of the two preceding Problems, will here be expounded by PB, $P\acute{B}$ &c. reſpectively: From whence it is alſo plain, that the Meaſures N, $\acute{N}$ &c. of the Angles whoſe Sines are $\frac{x \times Sin.\,\mathcal{Q}}{\sqrt{r^2 - 2brx + x^2}}$, $\frac{x \times Sin.\,\acute{\mathcal{Q}}}{\sqrt{r^2 - 2crx + x^2}}$ &c.* will here be expounded by OBP, $O\acute{B}P$, &c. &c.

Therefore the Fluent of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$, given in the Corollary to the foreſaid Propoſition, may be thus exhibited,

$$\frac{r^{m-n}}{n} \times \begin{cases} \overline{Sin.\ R} \times 2\,(OBP) - \overline{Co\text{-}ſ.\,R} \times 2\,(OA:PB) \\ \overline{Sin.\,3R} \times 2\,(O\acute{B}P) - \overline{Co\text{-}ſ.\,R} \times 2\,(OA:P\acute{B}) \\ \quad \&c. \qquad\qquad\qquad\qquad \&c. \end{cases}$$

Where the Arch R is $\left(= m \times \frac{180^\circ}{n}\right) = m \times AB$, and where $(OA:PB)$ is put (after the Manner of *Cotes*) to expreſs the hyperbolical Logarithm of $\frac{PB}{OA}$. It is alſo to be obſerved, *that*, when the laſt of the Points *B*,

* Art. 322. and 323.

$\overset{\prime}{B}$, $\overset{\prime\prime}{B}$ &c. falls upon H (which will always happen when n is an odd Number) the Angle, in the laſt Line of the Fluent, will vaniſh, and the correſponding Logarithm (which is that of $\frac{PH}{AO}$) muſt then be taken, inſtead of twice, only once.

In the very ſame Manner it will appear, *that*, the Arcs $\overset{\prime}{Q}$, $\overset{\prime\prime}{Q}$ &c. in the ſecond Caſe, where the Fluent of $\frac{x^{m-1}\dot{x}}{r^n - x^n}$ is ſought, will be, reſpectively, expounded by AC, $A\overset{\prime}{C}$ &c. alſo the correſponding Angles $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$ &c. by OCP, $O\overset{\prime}{C}P$ &c. and the Fluent itſelf by

$$\frac{r^{m-n}}{n}\times\left\{\begin{array}{l} \;^{*}\;\ldots\ldots\ldots - \ldots\ldots\;(OA:PC) \\ \overline{\text{Sin. }R}\times 2(OCP) - \overline{\text{Co-ſ. }R}\times 2(OA:PC) \\ \overline{\text{Sin.}2R}\times 2(O\overset{\prime}{C}P) - \overline{\text{Co-ſ.}2R}\times 2(OA:P\overset{\prime}{C}) \\ \quad\text{\&c.} \qquad\qquad\qquad\qquad\qquad \text{\&c.} \end{array}\right.$$

Where the Arch R $\left(= m \times \frac{360^\circ}{n}\right) = m \times AC$; and where, as well as in the preceding Caſe, all the Arcs, Sines and Co-ſines are ſuppoſed to have Unity for their Radius.

332. From the Fluents of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ and $\frac{x^{m-1}\dot{x}}{r^n - x^n}$, thus given, *thoſe* of $\frac{x^{vn+m-1}\dot{x}}{r^n + x^n}$, $\frac{x^{-vn+m-1}\dot{x}}{r^n + x^n}$, $\frac{x^{vn+m-1}\dot{x}}{r^n - x^n}$, and $\frac{x^{-vn+m-1}\dot{x}}{r^n - x^n}$, where v denotes any whole Number, may be very eaſily deduced; either from *Art.* 283. and 291. or (more readily) by dividing the Numerator by the Denominator, and continuing the

Quotient to as many Terms as there are Units in v *. By which means, if p be put $= vn + m$, $q = vn - m$, and the Fluents of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ and $\frac{x^{m-1}\dot{x}}{r^n - x^n}$ be denoted by V and W reſpectively, the Fluents, in the four Caſes ſpecified above, will be expreſſed by

$$\frac{x^{p-n}}{p-n} - \frac{r^n x^{p-2n}}{p-2n} + \frac{r^{2n} x^{p-3n}}{p-3n}\ (v) \pm r^{vn} V,$$

$$\frac{x^{-q}}{-qr^n} - \frac{x^{n-q}}{\overline{n-q}\,.\,r^{2n}} + \frac{x^{2n-q}}{\overline{2n-q}\,.\,r^{3n}}\ (v) \pm \frac{V}{r^{vn}},$$

$$-\frac{x^{p-n}}{p-n} - \frac{r^n x^{p-2n}}{p-2n} - \frac{r^{2n} x^{p-3n}}{p-3n}\ (v) + r^{vn} W,$$

and, $\frac{x^{-q}}{qr^n} + \frac{x^{n-q}}{\overline{n-q}\,.\,r^{2n}} + \frac{x^{2n-q}}{\overline{2n-q}\,.\,r^{3n}}\ (v) + \frac{W}{r^{vn}}$, reſpectively.

Moreover, from the ſame Fluents, *thoſe* of $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e+fz^q}$, and $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e-fz^q}$ will likewiſe become known:

For (having transformed the Fluxions here propoſed to $\frac{1}{e} \times \frac{z^{\frac{m}{n}q-1}\dot{z}}{1+\frac{fz^q}{e}}$, *&c.*) let $\frac{fz^q}{e}$ be put $= x^n$, or $x = \left.\frac{fz^q}{e}\right|^{\frac{1}{n}}$; then will $z^{\frac{m}{n}q} = \left.\frac{e}{f}\right|^{\frac{m}{n}} \times x^m$, and conſequently $\frac{mq}{n} \times z^{\frac{m}{n}q-1}\dot{z} = \left.\frac{e}{f}\right|^{\frac{m}{n}} \times mx^{m-1}\dot{x}$.

[*] Art. 150.

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Whence $z^{\frac{m}{n}q-1}\dot{z} = \frac{n}{q} \times \overline{\frac{e}{f}}\Big|^{\frac{m}{n}} \times x^{m-1}\dot{x}$, and $1 \pm \frac{fz^q}{e} = 1 \pm z^n$; and therefore $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e \pm fz^q} \left(= \frac{1}{e} \times \frac{n}{q} \times \overline{\frac{e}{f}}\Big|^{\frac{m}{n}} \times \frac{x^{m-1}\dot{x}}{1 \pm x^n} \right) = \frac{n}{qe} \times \overline{\frac{e}{f}}\Big|^{\frac{m}{n}} \times \frac{x^{m-1}\dot{x}}{1 \pm x^n}$: Whoſe Fluent is given, by *Prob.* 4. or 5. But, r being here $= 1$, the general Multiplicator $\frac{r^{m-n}}{n}$, there given, will be barely $= \frac{1}{n}$: Which, drawn into $\frac{n}{qe} \times \overline{\frac{e}{f}}\Big|^{\frac{m}{n}}$, gives $\frac{1}{qe} \times \overline{\frac{e}{f}}\Big|^{\frac{m}{n}}$, for the general Multiplicator in this Caſe.

One thing more, though well known to Mathematicians, it may be proper here to take notice of; and that relates to the Sines and Co-ſines of the fore-mention'd Arcs, R, $2R$, $3R$, *&c. &c.* (multiplying the ſeveral Angles and Ratios) ſome of which Arcs do frequently exceed the whole Periphery: When this happens to be the Caſe, the Periphery, or 360°, muſt be ſubſtracted as often as poſſible, and the Sine and Co ſine of the Remainder be taken. If the Remainder be greater than 180°, the Sine, falling in the lower Semi-Circle, will be negative; if, between 90° and 270°, the Co-ſine, falling beyond the Center, will be negative.

PROB. VI.

333. *To find the Fluent of* $\frac{x^{n+m-1}\dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}}$; *where* n *and* m *denote any whole poſitive Numbers, and where the given Expreſſion cannot be reſolved into two Binomials* (k *being leſs than Unity.* Art. 308. and 310.)

Let R be the Arch whoſe Coſine is k and Radius Unity, and let k' be the Sine of the ſame Arch; moreover, let the Arcs $\frac{R}{n}$, $\frac{R+360^\circ}{n}$, $\frac{R+2\times360^\circ}{n}$, $\frac{R+3\times360^\circ}{n}$ &c. be denoted by Q, Q', Q'', Q''', Q'''' &c. and let b', c', d' &c. and b, c, d &c. expreſs the Sines, and the Co-ſines of the ſame Arcs reſpectively.

Then will $\frac{nk'r^n x^n}{r^{2n}-2kr^n x^n+x^{2n}} = \frac{b'rx}{r^2-2brx+x^2} + \frac{c'rx}{r^2-2crx+x^2} + \frac{d'rx}{r^2-2drx+x^2}$ &c. (n) *by Lemma* 4.) From whence, multiplying the whole Equation by $\frac{x^{m-1}\dot{x}}{nk'r^n}$ we have $\frac{x^{n+m-1}\dot{x}}{r^{2n}-2kr^n x^n+x^{2n}} = \frac{1}{nk'r^{n-1}}$ into

$$\frac{b'x^m\dot{x}}{r^2-2brx+x^2} + \frac{c'x^m\dot{x}}{r^2-2crx+x^2} + \frac{d'x^m\dot{x}}{r^2-2drx+x^2} \text{ \&c.}$$

Now, the Fluent of the firſt Term hereof $\frac{b'x^m\dot{x}}{r^2-2brx+x^2}$ (if M be put for the hyp. Log. of $\frac{\sqrt{x^2-2brx+x^2}}{r}$, and N for the Arch whoſe Radius is Unity, and Sine $\frac{x\times\overline{\text{Sin. } Q}}{\sqrt{r^2-2brx+x^2}}$) will appear (*from Prop.* 2.) to be $=$

$$\overline{\text{Sin. } Q}\times\frac{x^{m-1}}{m-1} + \overline{\text{Sin. } 2Q}\times\frac{rx^{m-2}}{m-2} + \overline{\text{Sin. } 3Q}\times\frac{r^2x^{m-3}}{m-3} \text{ \&c. } (m-1) + r^{m-1}\times\overline{\text{Sin. } mQ\times M+\text{Co-ſin. } mQ\times N}.$$

From whence, if the Arcs whoſe Sines are $\frac{x\times\text{Sin. }\overset{\prime}{\mathcal{Q}}}{\sqrt{r^2-2crx+x^2}}$, $\frac{x\times\text{Sin. }\overset{\prime\prime}{\mathcal{Q}}}{\sqrt{r^2-2drx+x^2}}$ &c. be repreſented by $\overset{\prime}{M}$, $\overset{\prime\prime}{M}$ &c. and the Logarithms whoſe Numbers are $\frac{\sqrt{r^2-2crx+x^2}}{r}$, $\frac{\sqrt{r^2-2drx+x^2}}{r}$ &c. by $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$ &c. reſpectively, the Fluent of the whole Expreſſion, omitting the general Multiplicator $\left(\frac{1}{n\overset{\prime}{k}r^{n-1}}\right)$ will be

$$\left.\begin{array}{l}\text{Sin. }\mathcal{Q}\\ \text{Sin. }\overset{\prime}{\mathcal{Q}}\\ \text{Sin. }\overset{\prime\prime}{\mathcal{Q}}\\ \text{Sin. }\overset{\prime\prime\prime}{\mathcal{Q}}\\ \text{\&c.}\end{array}\right\}\times\frac{x^{m-1}}{m-1}+\left\{\begin{array}{l}\text{Sin. }2\mathcal{Q}\\ \text{Sin. }2\overset{\prime}{\mathcal{Q}}\\ \text{Sin. }2\overset{\prime\prime}{\mathcal{Q}}\\ \text{Sin. }2\overset{\prime\prime\prime}{\mathcal{Q}}\\ \text{\&c.}\end{array}\right\}\times\frac{rx^{m-2}}{m-2}+\left\{\begin{array}{l}\text{Sin. }3\mathcal{Q}\\ \text{Sin. }3\overset{\prime}{\mathcal{Q}}\\ \text{Sin. }3\overset{\prime\prime}{\mathcal{Q}}\\ \text{Sin. }3\overset{\prime\prime\prime}{\mathcal{Q}}\\ \text{\&c.}\end{array}\right\}$$

$$\times\frac{r^2x^{m-3}}{m-3}\quad(\text{\&c. to } m-1 \text{ Terms})$$

$$+\,r^{m-1}\times\left\{\begin{array}{l}\text{Sin. }m\mathcal{Q}\times M+\text{Co-ſ. }m\mathcal{Q}\times N\\ \text{Sin. }m\overset{\prime}{\mathcal{Q}}\times\overset{\prime}{M}+\text{Co-ſ. }m\overset{\prime}{\mathcal{Q}}\times\overset{\prime}{N}\\ \text{Sin. }m\overset{\prime\prime}{\mathcal{Q}}\times\overset{\prime\prime}{M}+\text{Co-ſ. }m\overset{\prime\prime}{\mathcal{Q}}\times\overset{\prime\prime}{N}\\ \text{Sin. }m\overset{\prime\prime\prime}{\mathcal{Q}}\times\overset{\prime\prime\prime}{M}+\text{Co-ſ. }m\overset{\prime\prime\prime}{\mathcal{Q}}\times\overset{\prime\prime\prime}{N}\\ \quad\text{\&c.}\qquad\qquad\text{\&c.}\end{array}\right\}$$

But, the Sines of the firſt Column being thoſe of an arithmetical Progreſſion (whoſe common Difference is

$\left.\frac{360^\circ}{n}\right)$ which ariſes by dividing the whole Periphery into n equal Parts, their Sum will, therefore, be equal to Nothing.

Moreover, the Sines of the ſecond Column, having $\frac{2\times 360^\circ}{n}$ for the common Difference of their reſpective Arcs do, alſo, divide the whole Periphery (twice taken) into n equal Parts, and therefore deſtroy each other.

The ſame is likewiſe true, with regard to the Sines of every other Column (except the laſt of all) when $m-1$ is leſs than n. But, if m be greater than n, the Arcs, in the Column, whoſe Place from the firſt, incluſive, is denoted by n, being expreſſed by $n\mathcal{Q}$, $n\overset{\prime}{\mathcal{Q}}$, $n\overset{\prime\prime}{\mathcal{Q}}$ *&c.* (or R, $R+360^\circ$, $R+2\times 360^\circ$ *&c.*) whereof the common Difference is the whole Periphery; the Sines of that Column do not deſtroy one another, but each is equal to that of the firſt Arc R (*Vid. Art.* 314. and 318.) and conſequently their Sum equal to $n\times Sin.\,R$.

In like Manner, if m be greater than $2n$, the Series, continued to $m-1$ Terms, will take in the Column, where the Arcs are $2n\mathcal{Q}$, $2n\overset{\prime}{\mathcal{Q}}$, $2n\overset{\prime\prime}{\mathcal{Q}}$ *&c.* (or $2R$, $2R+2\times 360^\circ$, $2R+4\times 360^\circ$ *&c.*) whereof the Sine of each is, alſo, equal to the Sine of the firſt ($2R$) and therefore their Sum $=n\times Sin.\,2R$.

Thus, alſo, it will appear that the Sines of the Column whoſe Diſtance from the firſt, incluſive, is $3n$ (when m is greater than $3n$) will be each equal to $Sin.\,3R$; *&c. &c.*

Therefore, ſeeing all the Columns do actually vaniſh, except thoſe above ſpecified; whoſe Places from the Beginning are denoted by n, $2n$, $3n$ *&c.* and whoſe correſponding Terms, or Multiplicators are, therefore, repreſented by $\frac{r^{n-1}x^{m-n}}{m-n}$, $\frac{r^{2n-1}x^{m-2n}}{m-2n}$, $\frac{r^{3n-1}x^{m-3n}}{m-3n}$ *&c.* it is evident that the whole Expreſſion will be reduced to

$$\overline{\text{Sin. } R} \times \frac{nr^{n-1}x^{m-n}}{m-n} + \overline{\text{Sin. } 2R} \times \frac{nr^{2n-1}x^{m-2n}}{m-2n}$$

$$+ \overline{\text{Sin. } 3R} \times \frac{nr^{3n-1}x^{m-3n}}{m-3n} \text{ \&c.}$$

$$+ r^{m-1} \text{ into } \begin{cases} \text{Sin. } mQ \times M + \text{Co-ſ.} mQ \times N \\ \text{Sin. } m\overset{\prime}{Q} \times \overset{\prime}{M} + \text{Co-ſ.} m\overset{\prime}{Q} \times \overset{\prime}{N} \\ \text{Sin. } m\overset{\prime\prime}{Q} \times \overset{\prime\prime}{M} + \text{Co-ſ.} m\overset{\prime\prime}{Q} \times \overset{\prime\prime}{N} \\ \text{Sin. } m\overset{\prime\prime\prime}{Q} \times \overset{\prime\prime\prime}{M} + \text{Co-ſ.} m\overset{\prime\prime\prime}{Q} \times \overset{\prime\prime\prime}{N} \\ \quad \text{\&c.} \qquad\qquad\qquad \text{\&c.} \end{cases}$$

Which, multiply'd by $\frac{1}{n\overset{\prime}{k}r^{n-1}}$, the foreſaid, general, Multiplicator, gives $\overline{\text{Sin. } R} \times \frac{x^{m-n}}{\overline{m-n}\,.\,k}$, $+ \overline{\text{Sin. } 2R} \times \frac{r^{n}x^{m-2n}}{\overline{m-2n}\,.\,\overset{\prime}{k}} + \overline{\text{Sin. } 3R} \times \frac{r^{2n}x^{m-3n}}{\overline{m-3n}\,.\,\overset{\prime}{k}}$ &c.

$$+ \frac{r^{m-n}}{n\overset{\prime}{k}} \times \begin{cases} \text{Sin. } mQ \times M + \text{Co-ſ. } mQ \times N \\ \text{Sin. } m\overset{\prime}{Q} \times \overset{\prime}{M} + \text{Co-ſ. } m\overset{\prime}{Q} \times \overset{\prime}{N} \\ \text{Sin. } m\overset{\prime\prime}{Q} \times \overset{\prime\prime}{M} + \text{Co-ſ. } m\overset{\prime\prime}{Q} \times \overset{\prime\prime}{N} \\ \text{Sin. } m\overset{\prime\prime\prime}{Q} \times \overset{\prime\prime\prime}{M} + \text{Co-ſ. } m\overset{\prime\prime\prime}{Q} \times \overset{\prime\prime\prime}{N} \\ \quad \text{\&c.} \qquad\qquad\qquad \text{\&c.} \end{cases}$$

for the true Fluent of $\frac{x^{n+m-1}\dot{x}}{r^{2n} - 2kr^{n}x^{n} + x^{2n}}$: Where the former Part of the Expreſſion muſt be continued to as many Terms as there are Units in $\frac{m-1}{n}$ (the Remainder, if any, being neglected.) *Q. E. I.*

Co-

COROLLARY.

334. If the Quotient ariſing from the Diviſion of m by n (when the former exceeds) be denoted by v, and the Remainder by t; or, which is the ſame, if $vn+t=m$, it is evident the Arcs $m\overset{\prime}{Q}$, $m\overset{\prime\prime}{Q}$, $m\overset{\prime\prime\prime}{Q}$ *&c.* which are reſpectively equal to $mQ+m\times\frac{360^\circ}{n}$, $mQ+2m\times\frac{360^\circ}{n}$, $mQ+3m\times\frac{360^\circ}{n}$, *&c.* (by Conſtruction) will alſo be equal to $mQ+v\times 360^\circ+t\times\frac{360^\circ}{n}$, $mQ+2v\times 360^\circ+2t\times\frac{360^\circ}{n}$ *&c.* whereof the Sines and Co-ſines (omitting $v\times 360^\circ$, $2v\times 360^\circ$ *&c.* the Multiples of the whole Periphery) are the ſame with thoſe of $mQ+t\times\frac{360^\circ}{n}$, $mQ+2t\times\frac{360^\circ}{n}$ *&c.* reſpectively.

Therefore, if the Arcs of the Progreſſion, whereof the firſt Term is mQ, and the common Difference $t\times\frac{360^\circ}{n}$, be repreſented by T, $\overset{\prime}{T}$, $\overset{\prime\prime}{T}$ *&c.* reſpectively; it follows that the Fluent of $\frac{x^{n+m-1}\dot{x}}{r^{2n}-2kr^nx^n+x^{2n}}$ $\left(\text{or, } \frac{x^{n+vn+t-1}\dot{x}}{r^{2n}-2kr^nx^n+x^{2n}}\right)$ will, *alſo*, be truly expreſſed by

$$\overline{Sin.\ R}\times\frac{x^{m-n}}{\overline{m-n}\,.\,\acute{k}}+\overline{Sin.\,2R}\times\frac{r^n x^{m-2n}}{\overline{m-2n}\,.\,\acute{k}}+\overline{Sin.\,3R}\times\frac{r^{2n}x^{m-3n}}{\overline{m-3n}\,.\,\acute{k}}\ \&c.\ \left(\frac{m-1}{n}\right)$$

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$$+\frac{r^{m-n}}{n\overset{\prime}{k}}\left\{\begin{array}{l}\text{Sin. } T\times M+\text{Co-ſ. } T\times N\\ \text{Sin. } \overset{\prime}{T}\times\overset{\prime}{M}+\text{Co-ſ. } \overset{\prime}{T}\times\overset{\prime}{N}\\ \text{Sin. } \overset{\prime\prime}{T}\times\overset{\prime\prime}{M}+\text{Co-ſ. } \overset{\prime\prime}{T}\times\overset{\prime\prime}{N}\\ \text{Sin. } \overset{\prime\prime\prime}{T}\times\overset{\prime\prime\prime}{M}+\text{Co-ſ. } \overset{\prime\prime\prime}{T}\times\overset{\prime\prime\prime}{N}\\ \quad \&c. \qquad\qquad \&c.\end{array}\right.$$

In the very ſame Manner the Fluent of $\frac{x^{n+m-1}}{r^{2n}+2kr^{n}x^{n}+x^{2n}}$ (where the Sign of the ſecond Term is poſitive) will be exhibited; if R be taken to denote the Arch whoſe Co-ſine is $-k$; which will, in this Caſe, be greater than a Quadrant.

PROPOSITION VII.

335. *To find the Fluent of* $\frac{x^{n-m-1}\dot{x}}{r^{2n}-2kr^{n}x^{n}+x^{2n}}$; *under the Reſtrictions mentioned in the laſt Problem.*

Let every thing remain as before: Then we ſhall have $\frac{x^{n-m-1}\dot{x}}{r^{2n}-2kr^{n}x^{n}+x^{2n}}=\frac{1}{n\overset{\prime}{k}r^{n-1}}$ into $\frac{\overset{\prime}{b}x^{-m}\dot{x}}{r^{2}-2brx+x^{2}}+\frac{\overset{\prime}{c}x^{-m}\dot{x}}{r^{2}-2crx+x^{2}}$ (n) Whereof the Fluent (*by Prob.* 3.) appears to be $\frac{1}{n\overset{\prime}{k}r^{n-1}}$ into

$$-\left\{\begin{array}{l}\text{Sin. } \mathcal{Q}\\ \text{Sin. } \overset{\prime}{\mathcal{Q}}\\ \text{Sin. } \overset{\prime\prime}{\mathcal{Q}}\\ \&c.\end{array}\right\}\times\frac{x^{1-m}}{\overline{m-1}\,.\,r^{2}}-\left\{\begin{array}{l}\text{Sin. } 2\mathcal{Q}\\ \text{Sin. } 2\overset{\prime}{\mathcal{Q}}\\ \text{Sin. } 2\overset{\prime\prime}{\mathcal{Q}}\\ \&c.\end{array}\right\}\times\frac{x^{2-m}}{\overline{m-2}\times r^{3}}$$

$$-\left\{\begin{matrix}\text{Sin. }3Q\\ \text{Sin. }3Q'\\ \text{Sin. }3Q'''\\ \&c.\end{matrix}\right\}\times\frac{x^{3-m}}{\overline{m-3}\cdot r^{4}}\ (m)+$$

$$\frac{1}{r^{m+1}}\times\left\{\begin{matrix}-\text{Sin. }mQ\times M+\text{Co-ſ. }mQ\times N\\ -\text{Sin. }mQ'\times M'+\text{Co-ſ. }mQ'\times N'\\ -\text{Sin. }mQ''\times M''+\text{Co-ſ. }mQ''\times N''\\ \&c. \qquad\qquad \&c.\end{matrix}\right.$$

Which, by Reaſoning as above, will be reduced to

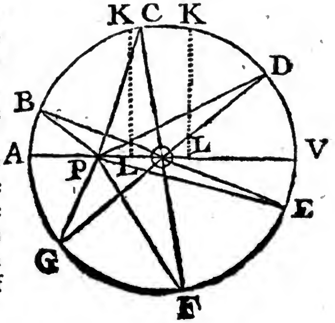
$$-\overline{\text{Sin. }R}\times\frac{x^{n-m}}{\overline{m-n}\cdot k'r^{2n}}-\overline{\text{Sin. }2R}\times\frac{x^{2n-m}}{\overline{m-2n}\cdot k'r^{3n}}$$

$$-\overline{\text{Sin. }3R}\times\frac{x^{3n-m}}{\overline{m-3n}\cdot k'r^{4n}}\left(\text{to }\frac{m}{n}\text{ Terms}\right)$$

$$+\frac{1}{k'nr^{n+m}}\times\left\{\begin{matrix}-\text{Sin. }T\times M+\text{Co-ſ. }T\times N\\ -\text{Sin. }T'\times M'+\text{Co-ſ. }T'\times N'\\ -\text{Sin. }T''\times M''+\text{Co-ſ. }T''\times N''\\ \&c. \qquad\qquad \&c.\end{matrix}\right.$$

Q. E. I.

SCHOLIUM.



336. If, from the Center O, of the Circle ABCD, whoſe Radius OA, or OV, is *r*, there be taken OL equal to *k* and OP=*x*; and if the Arch AB be to the Arch AK, whoſe Co-ſine is ± *k*, as 1 to *n*; and each

of the Arcs BC, CD, DE *&c.* be taken equal to $\frac{360°}{n}$ *&c. &c.* Then the Angles R, Q, $\overset{\prime}{Q}$ *&c.* ſpecified (in the two preceding Problems) being here expounded by AK, AB, AC *&c.* reſpectively, we have $PB = \sqrt{r^2 - 2brx + x^2}$, $PC = \sqrt{r^2 - 2crx + x^2}$ *&c.* (*Vid. Art.* 317. and 323.) Whence, alſo, the Angles N, $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$ *&c.* whoſe Sines are $\frac{x \times \text{Sin. } Q}{\sqrt{r^2 - 2brx + x^2}}$, $\frac{x \times \text{Sin. } \overset{\prime}{Q}}{\sqrt{r^2 - 2crx + x^2}}$, $\frac{x \times \text{Sin. } \overset{\prime\prime}{Q}}{\sqrt{r^2 - 2drx + x^2}}$ *&c.* will here be equal to B, C, D *&c.* Therefore the Fluents of $\frac{x^{n+m-1}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$, and $\frac{x^{n-m-1}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$ (there given) will, alſo, be truly defined by

$$\frac{x^{m-n}}{m-n} + \frac{\text{Sin. } 2R}{\text{Sin. } R} \times \frac{r^n x^{m-2n}}{m-2n} + \frac{\text{Sin. } 3R}{\text{Sin. } R} \times \frac{r^{2n} x^{m-3n}}{m-3n}$$

$$\left(\text{to } \frac{m-1}{n} \text{ Terms}\right)$$

$$+ \frac{r^{m-n}}{n \times \text{Sin. } R} \times \left\{ \begin{array}{ll} \text{Sin. } T \times (OB : PB) & + \text{Co-ſ. } T \times (B) \\ \text{Sin. } \overset{\prime}{T} \times (OC : PC) & + \text{Co-ſ. } \overset{\prime}{T} \times (C) \\ \text{Sin. } \overset{\prime\prime}{T} \times (OD : PD) & + \text{Co-ſ. } \overset{\prime\prime}{T} \times (D) \\ \text{Sin. } \overset{\prime\prime\prime}{T} \times (OE : PE) & + \text{Co-ſ. } \overset{\prime\prime\prime}{T} \times (-E) \\ \text{Sin. } \overset{\prime\prime\prime\prime}{T} \times (OF : PF) & + \text{Co-ſ. } \overset{\prime\prime\prime\prime}{T} \ (-F) \\ \&c. & \&c. \end{array} \right.$$

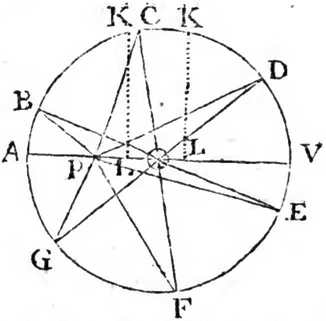
And

$$\text{And by} - \frac{x^{n-m}}{\overline{m-n}\,.\,r^{2n}} - \frac{\text{Sin. } 2R}{\text{Sin. } R} \times \frac{x^{2n-m}}{\overline{m-2n}\,.\,r^{3n}} -$$

$$- \frac{\text{Sin. } 3R}{\text{Sin. } R} \times \frac{x^{3n-m}}{\overline{m-3n}\,.\,r^{4n}} \left(\frac{m}{n}\right)$$

$$+ \frac{1}{nr^{n+m} \times \text{Sin.} R} \times \left\{ \begin{array}{l} -\text{Sin.} T \times (OB:PB) + \text{Co-ſ.} T \times (B) \\ -\text{Sin.} T' \times (OC:PC) + \text{Co-ſ.} T' \times (C) \\ -\text{Sin.} T'' \times (OD:PD) + \text{Co-ſ.} T'' \times (D) \\ -\text{Sin.} T''' \times (OE:PE) + \text{Co-ſ.} T''' \times (-E) \\ -\text{Sin.} T'''' \times (OF:PF) + \text{Co-ſ.} T'''' \times (-F) \\ \quad \&c. \end{array} \right.$$

reſpectively.



Where the Arc AK (or R) will be greater than a Quadrant when the Sign of k is poſitive; but leſs, when negative; and where the Arcs T, T', T'' &c. denote an arithmetical Progreſſion, whoſe firſt Term (T) is equal to $m \times AB$, and whereof the common Difference is equal to $\frac{360°}{n}$ (or BC) multiply'd by m, when m is leſs than n; but otherwiſe by the Remainder, of m divided by n.

337. Hence the Fluent of $\frac{z^{q\pm\frac{m}{n}q-1}\dot{z}}{e\mp fz^{q}+gz^{2q}}$, where q is any Number, either whole or broken, may be very eaſily deduced: For, having transform'd the Denominator to $g\times\overline{\frac{e}{g}\mp\frac{fz^{q}}{g}+z^{2q}}$, put $\frac{e}{g}=r^{2n}$, $\frac{f}{g}=2kr^{n}$, and $z^{q}=x^{n}$; and then it will become $=g\times\overline{r^{2n}\mp 2kr^{n}x^{n}+x^{2n}}$: Moreover, $z^{q\pm\frac{m}{n}q}$ being $=\overline{x^{n}}\big|^{1\pm\frac{m}{n}}=x^{n\pm m}$, and $\overline{q\pm\frac{m}{n}q}\times z^{q\pm\frac{m}{n}q-1}\dot{z}=\overline{n\pm m}\times x^{n\pm m-1}\dot{x}$, the Numerator will be reduced to $\frac{n}{q}\times x^{n\pm m-1}\dot{x}$: And ſo, we have $\frac{z^{q\pm\frac{m}{n}q-1}\dot{z}}{e\mp fz^{q}+gz^{2q}}=\frac{n}{qg}\times\frac{x^{n\pm m-1}\dot{x}}{r^{2n}\mp 2kr^{n}x^{n}+x^{2n}}$: In which $x=z^{\frac{q}{n}}$, $r=\overline{\frac{e}{g}}\Big|^{\frac{1}{2n}}$, and $k\ \left(=\frac{\frac{1}{2}f}{gr^{n}}\right)=\frac{\frac{1}{2}f}{\sqrt{eg}}$. But, it may be obſerv'd, that the Fluent hereof is, only, given when $\frac{\frac{1}{2}f}{\sqrt{eg}}$ (or its Equal k) is leſs than Unity *. Therefore, if $\frac{1}{2}f$ be greater than $\sqrt{eg}$; or if the Values of e and g are unlike, with regard to poſitive and negative, ſo that $\sqrt{eg}$ is impoſſible, the above Solution fails. But, here, the given Trinomial may be reſolved into two Binomials (*by. Art.* 310.) and, from thence, the Fluent may be found at two Operations (*by Prob.* 4. *and* 5.)

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* Art. 333.

For, by feigning $e \mp fy + gy^2 = 0$, in order to ſuch a Reſolution, we get $\frac{\pm\frac{1}{2}f+\sqrt{\frac{1}{4}f^2-eg}}{g}$, and $\frac{\pm\frac{1}{2}f-\sqrt{\frac{1}{4}f^2-eg}}{g}$ for the Roots of that Equation, or the two firſt Terms of the required Binomials: Which therefore are always poſſible when $\frac{1}{4}f^2-eg$ is poſitive, or when the foregoing Solution fails.

By denoting the ſaid Roots by H and K, the Trinomial $e \mp fz^q + gz^{2q}$ is reſolved into $g\times\overline{H-z^q}\times\overline{K-z^q}$, from whence $\frac{z^{q\pm\frac{m}{n}q-1}\dot{z}}{e \mp fz^q + gz^{2q}}$ is reduced to $\frac{z^{q\pm\frac{m}{n}q-1}\dot{z}}{g\times\overline{K-H}\times\overline{H-z^q}} + \frac{z^{q\pm\frac{m}{n}q-1}\dot{z}}{g\times\overline{H-K}\times\overline{K-z^q}}$, whoſe Fluent is given by *Art.* 332.

338. By proceeding the ſame Way the Fluent of $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e+fz^q+gz^{2q}+hz^{3q}}$ may likewiſe be found: For, ſince one, at leaſt, of the three Roots of the Equation $e+fy+gy^2+hy^3=0$, muſt be poſſible, the propoſed Fluxion, if it cannot be reſolved into three Binomials, may, however, be reduced to one Binomial and one Trinomial; and ſo, be brought under the foregoing Forms: But this being a Speculation too much out of the Way of common Uſe to be farther purſued, I ſhall here conclude this Section, with obſerving, that, when k, in the original Trinomial, above ſpecified, is neither leſs, nor greater than Unity, the Fluent cannot then be had directly, from either of the preceding Methods; but muſt be found by Compariſon from the Fluent of $\frac{x^{n\pm m-1}\dot{x}}{r^n \pm x^n}$. *Vid. Art.* 289.

SECTION VI.

The Manner of inveſtigating Fluents, when Quantities, and their Logarithms; Arcs and their Sines, &c. *are involved together: With other Caſes of the like Nature.*

PROB. I.

339. SUPPOSING *Q and n to denote given Quantities; 'tis propoſed to find the Fluent of* $x^n\dot{x}Q^x$.

Let $Q^x \times \overline{Ax^n + Bx^{n-1} + Cx^{n-2}}$ &c. be aſſumed for the Fluent required: Then the Fluxion thereof, which is

* Art. 252. $Q^x\dot{x} \times hyp.\ Log.\,Q\,{}^* \times \overline{Ax^n + Bx^{n-1} + Cx^{n-2}}$ &c. $+$ $Q^x \times \overline{n\dot{x}Ax^{n-1} + \overline{n-1}.B\dot{x}x^{n-2} + \overline{n-2}.C\dot{x}x^{n-3}}$ &c. muſt conſequently be $= x^n\dot{x}Q^x$: And therefore, by putting m for the *hyp. Log.* of Q, we have

$$\left.\begin{array}{l} mAx^n + mBx^{n-1} + mCx^{n-2} + mDx^{n-3}\ \&c. \\ -x^n + nAx^{n-1} + \overline{n-1}.Bx^{n-2} + \overline{n-2}.Cx^{n-3}\ \&c. \end{array}\right\} = 0$$

Whence, comparing the Coefficients of the homologous Terms, we get $A = \frac{1}{m}$, $B = -\frac{nA}{m} = -\frac{n}{m^2}$, $C = -\frac{\overline{n-1}.B}{m} = \frac{n.\overline{n-1}}{m^3}$ &c. and conſequently $Q^x \times \overline{Ax + Bx^{n-1} + Cx^{n-2} + \&c.} = \frac{Q^x}{m} \times \overline{x^n - \frac{nx^{n-1}}{m} + \frac{n.\overline{n-1}.x_{n-2}}{m^2} - \frac{n.\overline{n-1}.\overline{n-2}.x^{n-3}}{m^3}}$ &c. Which

Series, it is plain, will always terminate when n is a whole poſitive Number. *Q. E. I.*

340. In the preceding Problem the Coefficients A, B, C, *&c.* of the aſſumed Series were taken, in the common Way, as conſtant Quantities; which, becauſe of the general Multiplicator $\mathcal{Q}_x$, was ſufficient.

But, in other Caſes, where a proper Multiplicator, to expreſs the mechanical, or logarithmic, *&c.* Part of the required Fluent, cannot readily be known, it will be convenient to aſſume a Series for the *Whole* (independent of any general Multiplicator) wherein the Quantities A, B, C, D, *&c.* muſt be conſidered as variable.

PROB. II.

341. *To find the Fluent of* $z^m x^{n-1}\dot{x}$; z *being the Hyperbolic-Logarithm of* x; *and* m *and* n *any given Numbers.*

Let there be aſſumed $Az^m + Bz^{m-1} + Cz^{m-2} + Dz^{m-3}$ *&c.* = the Fluent of $z^m x^{n-1}\dot{x}$: Then, in Fluxions, we ſhall have

$$\left.\begin{array}{l}\dot{A}z^m + \dot{B}z^{m-1} + \dot{C}z^{m-2} + \dot{D}z^{m-3} \ \&c.\\ + mAz^{m-1}\dot{z} + \overline{m-1}.Bz^{m-2}\dot{z} + \overline{m-2}.Cz^{m-3}\dot{z}\&c.\end{array}\right\} = z^m x^{n-1}\dot{x}.$$

But $\dot{z} = \frac{\dot{x}}{x}$; whence, by ordering the Equation, there ariſes

$$\left.\begin{array}{c}\dot{A}\\ -x^{n-1}\dot{x}\end{array}\right\}\times z^m + \left.\begin{array}{c}\dot{B}\\ \frac{mA\dot{x}}{x}\end{array}\right\}\times z^{m-1} + \left.\begin{array}{c}\dot{C}\\ \frac{m-1.B\dot{x}}{x}\end{array}\right\}\begin{array}{c}z^{m-2}, \&c.\\ =0\end{array}$$

Now, by making the Coefficients of the like Powers of z, equal to Nothing, we have $\dot{A} = x^{n-1}\dot{x}$, $A = \frac{x^n}{n}$; $\dot{B} = \left(-\frac{mA\dot{x}}{x}\right) = -\frac{mx^{n-1}\dot{x}}{n}$, $B = -\frac{mx^n}{n^2}$;

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$\dot{C}\ (= -\frac{\overline{m-1}\,.\,B\dot{x}}{x} =)\ \frac{m\,.\,\overline{m-1}\,.\,x^{n-1}\dot{x}}{n^2}\quad C = \frac{m\,.\,\overline{m-1}\,.\,x^n}{n^3}$ *&c.* and conſequently the Fluent ſought $= \frac{x^n}{n}$ into $z^m - \frac{mz^{m-1}}{n} + \frac{m\;\overline{m-1}\,.\,z^{m-2}}{n^2} - \frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,z^{m-3}}{n^3} + \frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,\overline{m-3}\,.\,z^{m-4}}{n^4}$ *&c.* Which, when m is a whole poſitive Number, will terminate in $m+1$ Terms. *Q. E. I.*

PROB. III.

342. *To find the Fluent of* $z^n\dot{y}$; *z being the Arch of a given Circle, and y the Sine correſponding.*

Let there be aſſumed $Az^n + Bz^{n-1} + Cz^{n-2} + Dz^{n-3} =$ Fluent of $z^n\dot{y}$; then, by taking the Fluxion, we ſhall have

$$\left.\begin{array}{l} \dot{A}z^n + \dot{B}z^{n-1} + \dot{C}z^{n-2} + \dot{D}z^{n-3}\ \&c. \\ -z^n\dot{y} + nAz^{n-1}\dot{z} + \overline{n-1}\,.\,Bz^{n-2}\dot{z}\qquad \&c. \end{array}\right\} = 0$$

Whence, putting $\dot{A} - \dot{y} = 0$, $\dot{B} + nA\dot{z} = 0$, $\dot{C} + \overline{n-1}\,.\,B\dot{z} = 0$, $\dot{D} + \overline{n-2}\,.\,C\dot{z} = 0$, *&c.* we get $A = y$; $\dot{B} = -ny\dot{z}$, $\dot{C} = -\overline{n-1}\,.\,B\dot{z}$ *&c.*

But, if a and x be taken to denote the Radius and Co-ſine of the Arch z, it will appear, from *Art.* 142. that $y\dot{z} = -a\dot{x}$ and $x\dot{z} = a\dot{y}$: Therefore $\dot{B} = na\dot{x}$, and $B = nax$; alſo $\dot{C}\ (= -\overline{n-1}\,.\,B\dot{z}) = -n\,.\,\overline{n-1}\,.\,ax\dot{z} = -n\,.\,\overline{n-1}\,.\,a^2\dot{y}$, and $C = -n\,.\,\overline{n-1}\,.\,a^2y$; likewiſe $\dot{D}\ (= -\overline{n-2}\,.\,C\dot{z}) = n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,a^2y\dot{z} = -n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,a^3\dot{x}$, and $D = -n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,a^3x$ *&c.*

&c. *&c.* and conſequently $Az^n + Bz^{n-1} + Cz^{n-2}$ *&c.* $= yz^n + naxz^{n-1} - n . \overline{n-1} . a^2yz^{n-2} - n . \overline{n-1} . \overline{n-2} . a^3xz^{n-3} +$ *&c.*

$$= \left\{ \begin{array}{l} y \times \overline{z^n - n . \overline{n-1} . a^2z^{n-2} + n . \overline{n-1} . \overline{n-2} . \overline{n-3} . a^4z^{n-4}} \text{ \&c.} \\ x \times \overline{naz^{n-1} - n . \overline{n-1} . \overline{n-2} . a^3z^{n-3} + n . \overline{n-1} . \overline{n-2} . \overline{n-3} . \overline{n-4} . a^5z^{n-5}} \text{ \&c.} \end{array} \right.$$

Q. E. I.

In the very ſame Manner the Fluent of $z^n\dot{w}$, or $z^n \times -\dot{x}$ (w being the Verſed-Sine of the Arch z) will be found $= -xz^n + nyaz^{n-1} + n . \overline{n-1} . xa^2z^{n-2} - n . \overline{n-1} . \overline{n-2} . ya^3z^{n-3} - n . \overline{n-1} . \overline{n-2} . \overline{n-3} . xa^4z^{n-4} +$ *&c.*

PROB. IV.

343. *The Quantities, x, y and z being the ſame as in the preceding Problem; to find the Fluent of* $z^n x^r y^m \dot{y}$.

By aſſuming $Az^n + Bz^{n-1} + Cz^{n-2} + Dz^{n-3}$ *&c.* and proceeding as above, we have $\dot{A} = x^r y^m \dot{y}$, $\dot{B} = -nA\dot{z}$, $\dot{C} = -\overline{n-1}\,.\,B\dot{z}$, $\dot{D} = -\overline{n-2}\,.\,C\dot{z}$ *&c.* or (becauſe $\dot{z} = \frac{a\dot{y}}{x}$) $\dot{B} = -\frac{naA\dot{y}}{x}$, $\dot{C} = -\frac{\overline{n-1}\,.\,aB\dot{y}}{x}$, $\dot{D} = -\frac{\overline{n-2}\,.\,aC\dot{y}}{x}$ *&c.* Therefore, if the Fluent of $x^r y^m \dot{y}$ (found from *Art.* 142. and 291.) be denoted by Q; *that* of $\frac{Q\dot{y}}{x}$, by R; *that* of $\frac{R\dot{y}}{x}$, by S; *that* of $\frac{S\dot{y}}{x}$, by T *&c.* it follows that the Fluent of $z^n x^r y^m \dot{y}$ will be truly repreſented by $Qz^n - naRz^{n-1} + n.\overline{n-1}\,.\,a^2Sz^{n-2} - n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,a^3Tz^{n-3}$ *&c.*

COROLLARY

344. Since $\dot{y} = -\frac{x\dot{x}}{y} = \frac{x\dot{z}}{a}$ (*Vid. Art.* 142.) it follows that $z^n x^r y^m \dot{y}$ is $= -z^n x^{r+1} y^{m-1} \dot{x} = \frac{z^n x^{r+1} y^m \dot{z}}{a}$: Therefore the Fluents of theſe two laſt Expreſſions are, *alſo*, exhibited in the foregoing Series.

345. As the Values of Q, R, S *&c.* in the preceding Articles, are too complex to be purſued in a general Manner, it may not be amiſs to illuſtrate the Method of proceeding by an Example or two.

Let, then the Fluxion propoſed be $\frac{zy^2\dot{y}}{x}$: Where n being $=1$, $m=2$, and $r=-1$, we have $\dot{\mathcal{Q}}=\frac{y^2\dot{y}}{x}=\frac{y^2\dot{y}}{\sqrt{a^2-y^2}}$ (becauſe $\sqrt{a^2-y^2}=x$.) Whence $\mathcal{Q}=-\frac{1}{2}y\sqrt{a^2-y^2}+\frac{1}{2}az=-\frac{1}{2}yx+\frac{1}{2}az$ *, and therefore $\dot{R}\left(=\frac{\mathcal{Q}\dot{y}}{x}\right)=-\frac{1}{2}y\dot{y}+\frac{\frac{1}{2}az\dot{y}}{x}=-\frac{1}{2}y\dot{y}+\frac{1}{2}z\dot{z}$ (becauſe $\frac{a\dot{y}}{x}=\dot{z}$) and conſequently $R=-\frac{1}{4}y^2+\frac{1}{4}z^2$; and ſo, $\frac{az-yx}{2}\times z+a\times\frac{yy-zz}{4}$, or $\frac{az^2-2xyz+ay^2}{4}$, is the true Fluent of $\frac{zy^2\dot{y}}{x}$ $\left(=-zy\dot{x}=\frac{y^2z\dot{z}}{a}.\,*\right)$

* Art. 279.

* Art. 344.

Again, let the Fluent of $-p\dot{x}\times\overline{z+y}|^2$ (expreſſing the Content of the Solid generated by the Revolution of the *Cycloid*) be required.

Here, the given Expreſſion, in ſimple Terms, will become $-pz^2\dot{x}-2pzy\dot{x}-py^2\dot{x}$: Whereof the Fluent of the firſt Term $-pz^2\dot{x}$, will be had, by making $n=2$, $m-1=0$, and $r+1=0$ (*Vid. Form.* 2. *in Corol.*) Where, we therefore, have $\dot{\mathcal{Q}}=\frac{y\dot{y}}{x}=-\dot{x}$; whence $\mathcal{Q}=-x$; alſo $\dot{R}\left(\frac{\mathcal{Q}\dot{y}}{x}\right)=-\dot{y}$, and $R=-y$; likewiſe $\dot{S}\left(=\frac{R\dot{y}}{x}\right)=-\frac{y\dot{y}}{x}=\dot{x}$, $S=x$; and conſequently the Fluent of $-z^2\dot{x}$ $(\mathcal{Q}z^n-naRz^{n-1}+n\,.\,\overline{n-1}\,.\,a^2Sz^{n-2}$ &c.$)=-xz^2+2ayz+2a^2x$: To which, adding the Fluent $\left(\frac{az^2-2xyz+ay^2}{2}\right)$ of the ſecond Term $-2zy\dot{x}$ (found in the preceding Example) and alſo that of $-y^2\dot{x}$ (or $-a^2\dot{x}+x^2\dot{x}$, found the common Way) we get, in the Whole, $\frac{1}{2}\overline{a-x}\times z^2 + \overline{2ay-yx}\times z + \frac{1}{2}ay^2 + a^2x + \frac{1}{3}x^3$; which, multiply'd by p, and corrected, gives, p *into* $\frac{1}{2}\overline{a-x}\times z^2 + \overline{2ay-yx}\times z + \frac{1}{2}ay^2 + a^2x + \frac{1}{3}x^3 - \frac{4}{3}a^4$, for the true Fluent that was to be determined.

PROB. V.

346. *Suppoſing H to denote the Fluent of $\overline{k+lz^n}|^r \times z^{vn-1}\dot{z}$; to find the whole Fluent of $H\times\overline{a-bz^n}|^m \times z^{qn-1}\dot{z}$, (when $a-bz^n$ becomes equal to Nothing.)*

By reſolving $\overline{k+lz^n}|^r \times z^{vn-1}\dot{z}$ into ſimple Terms, and taking the Fluent, the ordinary Way, we get $H=$

$$\frac{k^r z^{vn}}{n}\times\overline{\frac{1}{v}+\frac{rlz^n}{\overline{v+1}\,.\,k}+\frac{r\,.\,\overline{r-1}\,.\,l^2z^{2n}}{2\,.\,\overline{v+2}\,.\,k^2}}\ \&c.$$

Which Value being ſubſtituted above, and p wrote inſtead of $q+v$, we ſhall have $H\times\overline{a-bz^n}|^m\times z^{qn-1}\dot{z} = \frac{k^r}{n}\times$

$$\overline{a-bz^n}|^m\times z^{pn-1}\dot{z}\ \textit{into}\ \frac{1}{v}+\frac{rlz^n}{\overline{v+1}\,.\,k}+\frac{r\,.\,\overline{r-1}\,.\,l^2z^{2n}}{2\,.\,\overline{v+2}\,.\,k^2}$$

$$+\frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,l^3z^{3n}}{2\,.\,3\,.\,\overline{v+3}\,.\,k^3}\ \&c.$$

Let, now, the Fluent of $\overline{a-bz^n}|^m\times z^{pn-1}\dot{z}$ (in the propoſed Circumſtance) be denoted by A, and put $t=p+m+1$; then it follows, from *Art.* 286. (by writing $\frac{1}{v}$ for e, $\frac{rl}{\overline{v+1}\,.\,k}$ for f, &c.) that $\frac{k^r}{n}\times A$ *into* $\frac{1}{v}+$

$$\frac{p \cdot r}{t \cdot \overline{v+1}} \times \frac{al}{bk} + \frac{p \cdot \overline{p+1} \cdot r \cdot \overline{r-1}}{t \cdot \overline{t+1} \cdot 2 \cdot \overline{v+2}} \times \overline{\frac{al}{bk}}\Big|^{2} +$$

$$\frac{p \cdot \overline{p+1} \cdot \overline{p+2} \cdot r \cdot \overline{r-1} \cdot \overline{r-2}}{t \cdot \overline{t+1} \cdot \overline{t+2} \cdot 2.3 \cdot \overline{v+3}} \times \overline{\frac{al}{bk}}\Big|^{3} + \text{\&c.}$$ will be the true Value of the Fluent. *Q. E. I.*

Note, p and $m+1$ muſt here be poſitive Quantities *; and it is alſo requiſite that $\frac{l}{k}$ ſhould be greater than $-\frac{b}{a}$; otherwiſe the Fluent will fail.

Ex. 1. *Let* $\dot{H} = \overline{1-y^2}\big|^{-\frac{1}{2}} \times \dot{y}$; *and let the whole Fluent of* $H \times \overline{1-y^2}\big|^{-\frac{1}{2}} \dot{y}$, *be demanded.*

Then, k being $=1$, $l = -1$, $z = y$, $n = 2$, $r = -\frac{1}{2}$, $v = \frac{1}{2}$; alſo $a=1$, $b=1$, $m = -\frac{1}{2}$, $q = \frac{1}{2}$; p $(=q+v)$ $=1$, t $(=p+m+1)$ $= \frac{3}{2}$, and A (= the *whole* Fluent of $\overline{1-y^2}\big|^{-\frac{1}{2}} y\dot{y}$) $= 1$; we ſhall, by ſubſtituting theſe ſeveral Values above, get $1 + \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 5} + \frac{1}{7 \cdot 7} + \frac{1}{9 \cdot 9} + \frac{1}{11.11}$ &c. $=$ Fluent of $H \times \overline{1-y^2}\big|^{-\frac{1}{2}} \times \dot{y}$ (or $H\dot{H}$) when $y=1$. Which Fluent being alſo expreſſed by $\frac{H^2}{2}$, it follows that $\frac{H^2}{2} = \frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81}$ &c. Where H is $\frac{1}{4}$ of the Periphery of the Circle whoſe Radius is Unity.

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* Art. 286.

Ex. 2. *Let* $\dot{H} = \overline{c^2 + z^2}|^{-\frac{3}{2}} \times \dot{z}$; *to find the Fluent of* $H \times \overline{b^2 - z^2}|^{-\frac{1}{2}} \times c^2\dot{z}$.

Here, $k = c^2$, $l = 1$, $n = 2$, $r = -\frac{3}{2}$, $v = \frac{1}{2}$; alſo $a = b^2$, $b = 1$, $m = -\frac{1}{2}$, $q = \frac{1}{2}$, $p\ (q + v) = 1$, $t\ (p + m + 1) = \frac{3}{2}$, and A (= *whole* Fluent of $\overline{b^2 - z^2}|^{-\frac{1}{2}} \times z\dot{z}) = b$: Whence, by Subſtitution, we have c^{-3} $b \times \overline{1 - \frac{1}{3} \times \frac{b^2}{c^2} + \frac{1}{5} \times \frac{b^4}{c^4} - \frac{1}{7} \times \frac{b^6}{c^6}}$ &c. which, multiply'd by c^2 (the Coefficient of $\dot{z}$) gives $\frac{1}{c} \times \overline{b - \frac{b^3}{3c^2} + \frac{b^5}{5c^4} - \frac{b^7}{7c^6}}$ &c. for the true Fluent in this Caſe: Where the Series is *that* expreſſing the Arch of the Circle whoſe Tangent is b and Radius c*; and is therefore equal to $c \times$ Arch, whoſe Radius is Unity and Tangent $= \frac{b}{c}$: Whence this laſt Arch (taken without the Multiplicator c) is the true Value of the Fluent.

* Art. 142.

SECTION VII.

Shewing how Fluents, found by Means of Infinite Serieſ's, are made to converge.

347. IT is found, *in Art.* 85. that the Fluent of $\overline{a + cz^n}|^m \times dz^{qn-1}\dot{z}$, in an Infinite Series, (making $m + q = s$) is expreſſed by $\frac{\overline{a + cz^n}|^{m+1} \times dz^{qn}}{qna} \times$

$\overline{1-\frac{\overline{s+1}.cz^n}{\overline{q+1}.a}+\frac{\overline{s+1}.\overline{s+2}.c^2z^{2n}}{\overline{q+1}.\overline{q+2}.a^2}}-$ &c. Whence it follows (and is evident by bare Inſpection) that the Fluent of $\overline{a-cy^n}|^{r}\times y^{qn-1}\dot{y}$ (where the ſecond Term under the *Vinculum* is negative) will be truly defined by $\frac{\overline{a-cy^n})^{r+1}\times y^{qn}}{qna}$ into $1+\frac{\overline{s+1}.cy^n}{\overline{q+1}.a}+\frac{\overline{s+1}.\overline{s+2}.c^2y^{2n}}{\overline{q+1}.\overline{q+2}.a^2}$. $+$ &c. ſuppoſing $s=r+q$.

But, beſides the Series here given, and Thoſe, *in Art.* 83. 84. expreſſing the ſame Value, the Fluent of $\overline{a-cy^n})^{r}\times y^{qn-1}\dot{y}$ will, yet, admit of another Form, different from all of them; by means whereof and *that* above, we ſhall be enabled to draw out ſome very uſeful Concluſions.

348. Put $z^n=\frac{ay^n}{a-cy^n}$; then $y^n=\frac{az^n}{a+cz^n}$, and therefore $ny^{n-1}\dot{y}=\frac{na^2z^{n-1}\dot{z}}{\overline{a+cz^n})^2}$; alſo $a-cy^n=\frac{a^2}{a+cz^n}$, and $y^{qn-1}\dot{y}$ $(=y^{qn-n}\times y^{n-1}\dot{y})=\frac{a^{q+1}z^{qn-1}\dot{z}}{\overline{a+cz^n})^{q+1}}$; and conſequently $\overline{a-cy^n})^{r}\times y^{qn-1}\dot{y}=a^{2r+q+1}\times\overline{a+cz^n})^{-r-q-1}\times z^{qn-1}\dot{z}$: Which Fluxion, ſo tranſform'd, being compared with $\overline{a+cz^n}^{m}\times dz^{qn-1}\dot{z}$; we have $m=-r-q-1$, $d=a^{2r+q+1}$, and s $(q+m)$ $=-r-1$; whence, by ſubſtituting theſe Values in the firſt Series, above given, the Fluent ſought will be had $=\frac{\overline{a+cz})^{-r-q}\times a^{2r+q}\times z^{qn}}{qn}\times\overline{1+\frac{rcz^n}{\overline{q+1}.a}+\frac{r.\overline{r-1}.c^2z^{2n}}{\overline{q+1}.\overline{q+2}.a^2}+\frac{r.\overline{r-1}.\overline{r-2}.c^3z^{3n}}{\overline{q+1}.\overline{q+2}.\overline{q+3}.a^3}}$ &c.

Which, by reſtoring y (or writing $\frac{a^2}{a-cy^n}$ and $\frac{ay^n}{a-cy^n}$ for their Equals $a+cz^n$, and z^n) becomes

$$\frac{\overline{a-cy^n}|^r \times y^{qn}}{qn} \times \overline{1+\frac{r}{q+1}\times\frac{cy^n}{a-cy^n}+\frac{r\cdot\overline{r-1}}{\overline{q+1}\cdot\overline{q+2}}\times\frac{c^2y^{2n}}{\overline{a-cy^n}|^2}}$$ &c. the true Fluent, of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$.

349. This Fluent may be otherwiſe found, independent of *that* above, in the following Manner:

It is evident, by taking the Fluxion of $\frac{\overline{a-cy^n}|^r \times y^{qn}}{qn}$ (which Quantity would be the Fluent ſought, if $\overline{a-cy^n}|^r$ was conſtant) that $\frac{\overline{a-cy^n}|^r \times y^{qn}}{qn}$ is $=$ the Fluent of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$ — Fluent of $\frac{rc}{q}\times \overline{a-cy^n}|^{r-1} \times y^{qn+n-1}\dot{y}$: This Equation, by tranſpoſing the laſt Term, and writing x in the room of $a-cy^n$ (for the Sake of Brevity) will become $Flu.\ x^r y^{qn-1}\dot{y} = \frac{x^r y^{qn}}{qn}+\frac{rc}{q}\times Flu.\ x^{r-1}y^{qn+n-1}\dot{y}$. From the very ſame Argument (if, inſtead of r, we ſubſtitute $r-1$, $r-2$ &c. ſucceſſively; and, for q. write $q+1$, $q+2$, $q+3$, &c. reſpectively) we ſhall, alſo, have

$$Flu.\ x^{r-1}y^{qn+n-1}\dot{y} = \frac{x^{r-1}y^{qn+n}}{\overline{q+1}\cdot n}+\frac{\overline{r-1}\cdot c}{q+1}\times Flu.\ x^{r-2}y^{qn+2n-1}\dot{y};$$

$$Flu.\ x^{r-2}y^{qn+2n-1}\dot{y} = \frac{x^{r-2}y^{qn+2n}}{\overline{q+2}\cdot n}+\frac{\overline{r-2}\cdot c}{q+2}\times Flu.\ x^{r-3}y^{qn+3n-1}\dot{y};$$

&c. &c.

Whence, by ſubſtituting theſe Values, one by one, in that of, *Flu.* $x^{r}y^{qn-1}\dot{y}$, we get

$$Flu.\ x^{r}y^{qn-1}\dot{y}=\frac{x^{r}y^{qn}}{qn}+\frac{rc}{q}\times\frac{x^{r-1}y^{qn+n}}{\overline{q+1}\,.\,n}+\frac{r.\overline{r-1}.c^{2}}{q.\overline{q+1}}$$

$$\times\ Flu.\ x^{r-2}y^{qn+2n-1}\dot{y}=\frac{x^{r}y^{qn}}{qn}+\frac{rcx^{r-1}y^{qn+n}}{q\,.\,\overline{q+1}\,.\,n}+$$

$$\frac{r\,.\,\overline{r-1}\,.\,c^{2}}{q\,.\,\overline{q+1}}\times\frac{x^{r-2}y^{qn+2n}}{\overline{q+2}\,.\,n}+\frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^{3}}{q\,.\,\overline{q+1}\,.\,\overline{q+2}}\times$$

$$Flu.\ x^{r-3}y^{qn+3n-1}\dot{y}=\frac{x^{r}y^{qn}}{qn}+\frac{rcx^{r-1}y^{qn+n}}{q\,.\,\overline{q+1}\,.\,n}+$$

$$\frac{r\,.\,\overline{r-1}\,.\,c^{2}x^{r-2}y^{qn+2n}}{q\,.\,\overline{q+1}\,.\,\overline{q+2}\,.\,n}+\frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^{3}x^{r-3}y^{qn+3n}}{q\,.\,\overline{q+1}\,.\,\overline{q+2}\,.\,\overline{q+3}\,.\,n}$$

&c. Where the Law of Continuation is manifeſt; and where, by making $\frac{x^{r}y^{qn}}{qn}$ a general Multiplicator, we ſhall have the very Series above exhibited.

350. From the Equality of the two foregoing Expreſſions, for the Fluent of $\overline{a-cy^{n}}^{\,r}\times y^{qn-1}\dot{y}$, (or $x^{r}y^{qn-1}\dot{y}$) the Buſineſs of finding Fluents, by infinite Serieſes, will, in many Caſes, be very much facilitated.

For, in the firſt Place, it follows (by dividing both by $\frac{\overline{a-c.^{n}}^{\,r+1}\times y^{qn}}{qna}$, or $\frac{x^{r+1}y^{qn}}{qna}$) that the Serieſes $1+\frac{\overline{s+1}\,.\,cy^{n}}{\overline{q+1}\,.\,a}+\frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^{2}y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,a^{2}}$ &c. and $\frac{a}{x}\times$

$$\frac{\qquad}{1+\frac{rcy^{n}}{\overline{q+1}\,.\,x}+\frac{r\,.\,\overline{r-1}\,.\,c^{2}y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,x^{2}}+\frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^{3}y^{3n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,\overline{q+3}\,.\,x^{3}}+}$$

&c. muſt alſo be equal to each other, let the ſeveral

Quantities, therein concerned, be what they will (which may be otherwiſe proved, independent of Fluxions.) Therefore, if in the room of q and s we write any other Quantities p and t, the Equation will, *ſtill*, hold, and will then become $1+\frac{\overline{t+1}\cdot cy^n}{\overline{p+1}\cdot a}+\frac{\overline{t+1}\cdot\overline{t+2}\cdot c^2y^{2n}}{\overline{p+1}\cdot\overline{p+2}\cdot a^2}$ $+\ \&c. = \frac{a}{x}\times\overline{1+\frac{rcy^n}{\overline{p+1}\cdot x}+\frac{r\cdot\overline{r-1}\cdot c^2y^{2n}}{\overline{p+1}\cdot\overline{p+2}\cdot x^2}}\ \&c.$ (t being $=p+r$.)

Moreover, if as many Terms of the firſt Series $1+\frac{\overline{s+1}\cdot cy^n}{\overline{q+1}\cdot a}+\frac{\overline{s+1}\cdot\overline{s+2}\cdot c^2y^{2n}}{\overline{q+1}\cdot\overline{q+2}\cdot a^2}+\frac{\overline{s+1}\cdot\overline{s+2}\cdot\overline{s+3}\cdot c^3y^3}{\overline{q+1}\cdot\overline{q+2}\cdot\overline{q+3}\cdot a^3}$ &c. be taken as are denoted by any given Number v, and the laſt of them be repreſented by Q, it is evident, from the Law of the Series, that the firſt of the remaining Terms will be expreſſed by $Q\times\frac{s+v}{q+v}\times\frac{cy^n}{a}$; the ſecond, of them, by $Q\times\frac{\overline{s+v}}{q+v}\times\frac{\overline{s+v+1}}{q+v+1}\times\frac{c^2y^{2n}}{a^2}$ &c. and therefore the Sum of all of them (putting $q+v=p$ and $s+v$ $(=r+q+v)=t$) will be $=Q\times\frac{t}{p}\times\frac{cy^n}{a}+Q\times\frac{t}{p}\times\frac{t+1}{p+1}\times\frac{c^2y^{2n}}{a^2}+\ \&c. =$ $\frac{tQcy^n}{pa}\times\overline{1+\frac{\overline{t+1}\cdot cy^n}{\overline{p+1}\cdot a}+\frac{\overline{t+1}\cdot\overline{t+2}\cdot c^2y^{2n}}{\overline{p+1}\cdot\overline{p+2}\cdot a^2}+\ \&c.}$ $=\frac{tQcy^n}{px}\times\overline{1+\frac{rcy^n}{\overline{p+1}\cdot x}+\frac{r\cdot\overline{r-1}\cdot c^2y^{2n}}{\overline{p+1}\cdot\overline{p+2}\cdot x^2}}\ \&c.$ (by writing the Series found above in the room of its Equal) and conſequently the whole Series (including the v firſt Terms) $=1+\frac{\overline{s+1}\cdot cy^n}{\overline{q+1}\cdot a}+$

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$$\frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,a^2}\,(v) + \frac{t\mathcal{Q}cy^n}{px} \times \overline{1 + \frac{rcy^n}{\overline{p+1}\,.\,x} +}$$

$$\overline{\frac{r\,.\,\overline{r-1}\,.\,c^2y^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,x^2} + \frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^3y^{3n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,\overline{p+3}\,.\,x^3}} + \text{\&c.}$$

Which, drawn into the general Multiplicator $\frac{x^{r+1} \times y^{qn}}{qna}$ (*vid. Art.* 347.) will give the Fluent of $\overline{a-cy^n}|^r\,xy^{qn-1}\dot{y}$ (or $x^r y^{qn-1}\dot{y}$) according to a new Form, compounded out of the two preceding ones; where the ſecond Series (the Value of p being large in reſpect of r) will always converge much faſter than the remaining Part of the firſt, for which it is ſubſtituted: But this will, more fully, appear from what follows hereafter. It will be proper to take notice here that the Fluent of $\overline{a+cz^n}|^m \times z^{qn-1}\dot{z}$ (the Fluxion firſt propoſed, where the ſecond Term under the Vinculum is poſitive) will alſo be had from hence (by writing z for y, m for r, and $-c$ for c) and is therefore equal to $\frac{x^{m+1}z^{qn}}{qna}$ drawn into the Sum of the two following Serieſes,

$$1 - \frac{\overline{s+1}\,.\,cz^n}{\overline{q+1}\,.\,a} + \frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2z^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,a^2} - \frac{\overline{s+1}\,.\,\overline{s+2}\,.\,\overline{s+3}\,.\,c^3z^{3n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,\overline{q+3}\,.\,a^3}\,(v)$$

$$- \frac{t\mathcal{Q}cz^n}{px} \times \overline{1 - \frac{mcz^n}{\overline{p+1}\,.\,x} + \frac{m\,.\,\overline{m-1}\,.\,c^2z^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,x^2} -}$$

$$\overline{\frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,c^3z^{3n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,\overline{p+3}\,.\,x^3}} + \text{\&c.}$$

Where, $s=m+q$, $p=v+q$, $t=s+v$, $x=a+cz^n$, and $\mathcal{Q}=$ the laſt Term of the firſt Series continued to v Terms, v being any whole Number, at pleaſure. A few Examples will ſhew the Uſe of what is above delivered.

351. Ex. 1. *Let* $\frac{\dot{z}}{1+z}$, or $\overline{1+z}|^{-1}\dot{z}$, *be propounded.*

Which being compared with $\overline{a+cz^n}|^{m}\times z^{qn-1}\dot{z}$, we have $a=1$, $c=1$, $n=1$, $x=1+z$, $m=-1$, $qn-1=0$, or $q=1$; whence alſo $s\ (m+q)=0$, $p\ (v+q)=v+1$, $t\ (s+v)=v$, and conſequently the Fluent itſelf (by ſubſtituting theſe ſeveral Values in the laſt general Theorem) $= z$ into $1-\frac{z}{2}+\frac{z^2}{3}-\frac{z^3}{4}\ (v)-\frac{vzQ}{\overline{v+1}.x}$ $\times\ \overline{1+\frac{z}{\overline{v+2}.x}+\frac{2.z^2}{\overline{v+2}.\overline{v+3}.x^2}+\frac{2.3.z^3}{\overline{v+2}.\overline{v+3}.\overline{v+4}.x^3}}$ &c. Where (Q) the laſt Term of the firſt Series being $\pm\frac{z^{v-1}}{v}$, the Multiplicator $\left(\frac{vzQ}{\overline{v+1}.x}\right)$ to the Second, will be $=\mp\frac{z^{v}}{\overline{v+1}.x}$; and ſo the Fluent itſelf will be reduced to $z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}\ (v)\mp\frac{z^{v+1}}{\overline{v+1}.x}\times$ $\overline{1+\frac{z}{\overline{v+2}.x}+\frac{2.z^2}{\overline{v+2}.\overline{v+3}.x^2}}+$ &c. In which the Signs − and +, before z^{v+1}, obtain alternately, according as v is an odd or even Number. But, to ſhew the Advantage of expreſſing the Fluent in this Manner, by two different Serieſes, let $z=1$, and let v be taken $=8$; then the Value of the firſt Series (continued to 8 Terms) being $=0{,}6345238$ &c. and *That* of the ſecond Series $=\frac{1}{18}+\frac{A}{20}+\frac{2B}{22}+\frac{3C}{24}+\frac{4D}{26}+\frac{5E}{28}$ &c. (where A, B, C, D &c. denote the Terms preceding thoſe where they ſtand) $=0{,}0555555+0{,}0027778+0{,}0002525+0{,}0000316+0{,}0000048+0{,}0000009+0{,}0000002=0{,}0586233$; it is evident that the Fluent of $\frac{\dot{z}}{1+z}$, when z becomes $=1$, will be $=0{,}6345238+0{,}0586233=0{,}6931471$: Which is true to the very laſt Decimal Place; and would have required, at leaſt, 100000 Terms of the firſt, or common, Series.

352. Ex. 2. *Let the Fluent of* $\frac{\dot{z}}{1+z^2}$ *(expreſſing the Arch whoſe Radius is* 1 *and Tangent* z*) be required.*

In this Caſe we have $a=1$, $c=1$, $n=2$, $x=1+zz$, $m=-1$, $qn-1=0$, or $q=\frac{1}{2}$, $s=-\frac{1}{2}$, $p=v+\frac{1}{2}$, and the Fluent itſelf $=z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}\ (v)\ \pm$

$$\frac{z^{2v+1}}{\overline{2v+1}\cdot x}\times\overline{1+\frac{2\cdot z^2}{\overline{2v+3}\cdot x}+\frac{2\cdot 4\cdot z^4}{\overline{2v+3}\cdot\overline{2v+5}\cdot x^2}+\frac{2\cdot 4\cdot 6\cdot z^6}{\overline{2v+3}\cdot\overline{2v+5}\cdot\overline{2v+7}\cdot x^3}}\ \&c.$$

Where, if z be taken $=1$, and $v=6$, we ſhall have

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{26}\times\overline{1+\frac{1}{15}+\frac{1}{15}\times\frac{2}{17}+\frac{1}{15}\times\frac{2}{17}\times\frac{3}{19}}\ \&c.=0{,}785398$$

$=$ the Fluent of $\frac{\dot{z}}{1+z^2}$ when $z=1$ ($=\frac{1}{4}$ of the Periphery of the foreſaid Circle) Which Number, brought out by taking, only, 8 Terms of the ſecond Series, is more exact than if 100000 Terms of the common Series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}$ &c. had been uſed. And, if z be taken $=\sqrt{\frac{1}{3}}$ (= Tangent of 30°) and $v=6$, as before, the ſame Number of Terms will be ſufficient to give the Anſwer, true to twice the Decimal Places above exhibited.

353. Ex. 3. *Let the Fluxion propoſed be* $\overline{e^4+y^4}^{\frac{1}{2}} \times \dot{y}$.

Here we have, $a=e^4$, $c=1$, $z=y$, $n=4$, $x=e^4+y^4$, $m=\frac{1}{2}$, $q=\frac{1}{4}$, $s\ (m+q)=\frac{3}{4}$, $p\ (v+q)=v+\frac{1}{4}$; $t\ (s+v)=v+\frac{3}{4}$; and therefore the Fluent ſought (by Subſtitution) is $= \frac{x^{\frac{3}{2}}y}{e^4}$ into

$$1-\frac{7y^4}{5e^4}+\frac{7\cdot 11y^8}{5\cdot 9e^8}-\frac{7\cdot 11\cdot 15y^{12}}{5\cdot 9\cdot 13e^{12}}\ (v)-\frac{\overline{4v+3}\cdot\mathcal{Q}y^4}{\overline{4v+1}\cdot x}\times\overline{1-\frac{2y^4}{\overline{4v+5}\cdot x}-\frac{2\cdot 2y^8}{\overline{4v+5}\cdot\overline{4v+9}\cdot x^2}-\frac{2\cdot 6\cdot 2y^{12}}{\overline{4v+5}\cdot\overline{4v+9}\cdot\overline{4v+13}\cdot x^3}-}$$

&c. In which (as in all other Caſes) $\mathcal{Q}$ denotes the laſt Term of the firſt Series. This Fluent approximates equally faſt with thoſe in the foregoing Examples: And it may be obſerved farther, that the Fluent will always converge, however great the Value of z is taken, if both a and c, in the general Fluxion $\overline{a+cz^n}^{m}\times z^{qn-1}\dot{z}$, are poſitive Quantities. But, if the ſecond Term under the Vinculum be negative, the Caſe will be otherwiſe, when that Term becomes greater than half the Firſt; ſince the Powers of $\frac{cz^n}{x}$, in the latter Part of the Fluent, will then form an increaſing Geometrical Progreſſion. It may, therefore, be of uſe to ſhew how the Theorem may be varied ſo as to anſwer in this Caſe. In order thereto, if in the Equations $s=r+q$, and

$$1+\frac{\overline{s+1}\cdot cy^n}{\overline{q+1}\cdot a}+\frac{\overline{s+1}\cdot\overline{s+2}\cdot c^2y^{2n}}{\overline{q+1}\cdot\overline{q+2}\cdot a^2}\ \&c. = \frac{a}{x}\times\overline{1+\frac{rcy^n}{\overline{q+1}\cdot x}+\frac{r\cdot\overline{r-1}\cdot c^2y^{2n}}{\overline{q+1}\cdot\overline{q+2}\cdot x^2}}\ \&c.$$

(*given in Art.* 350.) you write k for r, and p for q, and multiply by $\frac{x}{a}$, you will have $s=k+p$, and $1+\frac{kcy^n}{\overline{p+1}.x}+\frac{k.\overline{k-1}.c^2y^{2n}}{\overline{p+1}.\overline{p+2}.x^2}$ &c. $=\frac{x}{a}\times\overline{1+\frac{\overline{s+1}.cy^n}{\overline{p+1}.a}+\frac{\overline{s+1}.\overline{s+2}.c^2y^{2n}}{\overline{p+1}.\overline{p+2}.a^2}}$ &c.

Moreover, if the v firſt Terms of the above Series $1+\frac{rcy^n}{\overline{q+1}.x}+\frac{r.\overline{r-1}.c^2y^{2n}}{\overline{q+1}.\overline{q+2}.x^2}$ &c. be taken, and the laſt of them be denoted by Q, it is plain the firſt of the remaining Terms will be $=Q\times\frac{r-v+1}{q+v}\times\frac{cy^n}{x}$, the ſecond $=Q\times\frac{r-v+1}{q+v}\times\frac{r-v}{q+v+1}\times\frac{c^2y^{2n}}{x^2}$, &c. and the Sum of them all (putting $q+v=p$, and $r-v=k$) equal to $\frac{\overline{k+1}.Qcy^n}{px}\times\overline{1+\frac{kcy^n}{\overline{p+1}.x}+\frac{k.\overline{k-1}.c^2y^{2n}}{\overline{p+1}.\overline{p+2}.x^2}}$ &c. $=\frac{\overline{k+1}.Qcy^n}{px}\times\frac{x}{a}\times\overline{1+\frac{\overline{s+1}.cy^n}{\overline{p+1}.a}+\frac{\overline{s+1}.\overline{s+2}.c^2y^{2n}}{\overline{p+1}.\overline{p+2}.a^2}}$ &c. (by the Equation above) and conſequently the Sum of the whole Series ($1+\frac{rcy^n}{\overline{q+1}.x}$ &c.) $=1+\frac{rcy^n}{\overline{q+1}.x}+\frac{r.\overline{r-1}.c^2y^{2n}}{\overline{q+1}.\overline{q+2}.x^2}+\frac{r.\overline{r-1}.\overline{r-2}.c^3y^{3n}}{\overline{q+1}.\overline{q+2}.\overline{q+3}.x^3}$ (v) $+\frac{\overline{k+1}\times cy^nQ}{pa}\times\overline{1+\frac{\overline{s+1}.cy^n}{\overline{p+1}.a}+\frac{\overline{s+1}.\overline{s+2}.c^2y^{2n}}{\overline{p+1}.\overline{p+2}.a^2}+}$ &c. Which, multiply'd by $\frac{x^r y^{qn}}{qn}$, gives the Fluent of $\overline{a-cy^n}^r$ $\times y^{qn-1}\dot{y}$ (* or $x^r y^{qn-1}\dot{y}$) where $k=r-v$, $p=v+q$, s $(=k+p)=r+q$ and $x=a-cy^n$. I ſhall put down one Example of the Uſe of this laſt general Expreſſion; where we will take $\dot{y}\sqrt{2y-y^2}$, or $\overline{2-y}^{\frac{1}{2}}\times y^{\frac{1}{2}}\dot{y}$ (being the Fluxion of the Area of the Circle whoſe Radius is Unity and verſed Sine y) In which Caſe, $a=2$, $c=1$, $n=1$, $r=\frac{1}{2}$, $qn-1=\frac{1}{2}$, or $q=\frac{3}{2}$, $k=-v+\frac{1}{2}$, $p=v+\frac{3}{2}$, $s=2$, $x=2-y$; and therefore the Fluent ſought $=\frac{2x^{\frac{1}{2}}y^{\frac{3}{2}}}{3}$ into $1+\frac{y}{5x}-\frac{y^2}{5\cdot 7x^2}+\frac{3y^3}{5\cdot 7\cdot 9x^3}-\frac{3y^4}{7\cdot 9\cdot 11x^4}+\frac{3y^5}{9\cdot 11\cdot 13x^5}$ (v) $\mp\frac{\overline{2v-3}\cdot yQ}{\overline{2v+3}\cdot 2}\times\overline{1+\frac{3y}{2v+5}+\frac{3\cdot 4y^2}{\overline{2v+5}\cdot\overline{2v+7}}+\frac{3\cdot 4\cdot 5y^3}{\overline{2v+5}\cdot\overline{2v+7}\cdot\overline{2v+9}}}$ &c. Which, if y be taken $=1$, and $v=5$, will become $=\frac{2}{3}+\frac{A}{5}-\frac{B}{7}+\frac{C}{3}-\frac{5D}{11}+\frac{7E}{2\times 13}+\frac{3F}{15}+\frac{4G}{17}+\frac{5H}{19}$ &c. $=0,785398$ (where A, B, C &c. denote the ſeveral Terms, reſpectively, without their Signs.) In bringing out which Concluſion, ſix Terms of the ſecond Series are required: But if y be taken $=\frac{1}{2}$ the Radius of the foreſaid Circle, then four Terms of each Series will be more than ſufficient to give the ſame Number of Decimal Places. And it may likewiſe be obſerved, that, although no general Rule can be laid down for aſſigning the Value of v, ſo as to anſwer the beſt in all Caſes, yet the Concluſion will,

* Art. 348, 349.

will, for the general Part, require the feweſt Terms, when the Number of thoſe, taken in each Series, is nearly the ſame.

354. But, after all, another Theorem or Series, ſtill, ſeems wanting, to expreſs the Value of the whole Fluent, when the Quantity under the *Vinculum* becomes equal to Nothing (which, in the Reſolution of Problems, is, commonly, what is required.) For, it is plain the laſt, above given, anſwers no better, *here*, than that preceding it; becauſe (the Diviſor (x) being Nothing) the former Part of it fails.

In order, therefore, to determine a proper *Form*, to obtain in this Circumſtance, it will be requiſite to obſerve, firſt of all, *from Article* 286. that the whole Fluent of $\overline{a-bz^n}\rceil^{m}\times z^{pn+vn-1}\dot{z}$, ſuppoſing *that* of $\overline{a-bz^n}\rceil^{m}\times z^{pn-1}\dot{z}$ to be denoted by A, will be truly expreſſed by $\frac{p}{t}\times\frac{p+1}{t+1}\times\frac{p+2}{t+2}\,(v)\times\frac{a^{v}A}{b^{v}}$: In which $t=m+p+1$; and where it is requiſite that the Values of $m+1$ and p ſhould be poſitive, otherwiſe, A being infinite, the Fluent (or Compariſon) fails. Hence, becauſe the whole Fluent of $\overline{a-bz^n}\rceil^{m}\times z^{n-1}\dot{z}$, (when $a-bz^n=0$) is found $=\frac{a^{m+1}}{\overline{m+1}\times nb}$, by the common Way*, it follows, by writing this Value in the Room of A, and expounding p by 1, that the *whole Fluent* of $\overline{a-bz^n}\rceil^{m}\times z^{n+vn-1}\dot{z}$ is rightly expreſſed by $\frac{1}{m+2}\times\frac{2}{m+3}\times\frac{3}{m+4}\,(v)\times\frac{a^{m+v+1}}{\overline{m+1}\times nb^{v+1}}$, or by $\frac{1}{m+1}\times\frac{2}{m+2}\times\frac{3}{m+3}\,(v+1)\times\frac{a^{m+v+1}}{\overline{v+1}\times nb^{v+1}}$: Whence

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* Art. 77. and 78.

That of $\overline{a-bz^n}|^m \times z^{rn-1}\dot{z}$, by ſubſtituting r inſtead of $v+1$, will conſequently be equal to $\frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}$ (r) $\times \frac{a^{m+r}}{rnb^r}$. Let this Quantity be denoted by B; then, *by the ſame Article*, the Fluents of the ſeveral Terms of the Series 1, $\frac{bz^n}{a}$, $\frac{b^2z^{2n}}{a^2}$, $\frac{b^3z^{3n}}{a^3}$ *&c.* drawn into the general Multiplicator $\overline{a-bz^n}|^m \times z^{rn-1}\dot{z}$, will be, reſpectively, expounded by thoſe of the Series 1, $\frac{r}{t}$, $\frac{r.\overline{r+1}}{t.\overline{t+1}}$, $\frac{r.\overline{r+1}.\overline{r+2}}{t.\overline{t+1}.\overline{t+2}}$ *&c.* drawn into B; t being $=m+r+1$.

If now the Differences of the Quantities 1, $\frac{r}{t}$, $\frac{r.\overline{r+1}}{t.\overline{t+1}}$ *&c.* be, continually, taken †; and for $r-t$ its Equal $-m-1$ be ſubſtituted, the Value of any Term of the Series, whoſe Diſtance from the firſt, excluſive, is denoted by s, or whoſe correſponding Term, in the preceding Series, is $\frac{b^s z^{sn}}{a^s}$, will be univerſally expreſſed by $1 - \frac{s.\overline{m+1}}{1.t} + \frac{s.\overline{s-1}.\overline{m+1}.\overline{m+2}}{1.2 \times t.\overline{t+1}} - \frac{s.\overline{s-1}.\overline{s-2}.\overline{m+1}.\overline{m+2}.\overline{m+3}}{1.2.3 \times t.\overline{t+1}.\overline{t+2}} +$ *&c.* Where, if s be interpreted by 0, 1, 2, 3 *&c.* ſucceſſively, you will have the Values 1, $\frac{r}{t}$, $\frac{r.\overline{r+1}}{t.\overline{t+1}}$ *&c.* above exhibited: But, if s be taken as a Fraction, then the Value of ſuch an intermediate Term will be found as will give the Fluent of $\frac{b^s z^{sn}}{a^s} \times \overline{a - bz^n}|^m \times z^{rn-1}\dot{z}$, in any propoſed Circumſtance of s; which Fluent, it is evident, will therefore be expreſſed by $B \times \overline{1 - \frac{s \cdot \overline{m+1}}{1 \cdot t} + \frac{s \cdot \overline{s-1} \cdot \overline{m+1} \cdot \overline{m+2}}{1 \cdot 2 \cdot t \cdot \overline{t+1}}}$ &c. or its Equal $\frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}\ (r) \times \frac{a^{m+r}}{rnb^r}$ into $1 - \frac{s \cdot \overline{m+1}}{1 \cdot t} - \frac{\overline{s-1} \cdot \overline{m+2}}{2 \cdot \overline{t+1}} \times E - \frac{\overline{s-2} \cdot \overline{m+3}}{3 \cdot \overline{t+2}} \times F - \frac{\overline{s-3} \cdot \overline{m+4}}{4 \cdot \overline{t+3}} \times G$ &c. (where E, F, G &c. denote the Terms immediately preceding thoſe where they ſtand, under their proper Signs.) Whence, dividing by $\frac{b^s}{a^s}$, we have $\frac{1}{m+1} \times \frac{2}{m+2}\ (r) \times \frac{a^{m+r+s}}{rnb^{r+s}} \times \overline{1 - \frac{s \cdot \overline{m+1}}{t} - \frac{\overline{s-1} \cdot \overline{m+2}}{2 \cdot \overline{t+1}} \times E}$, &c. for the true Fluent of $\overline{a - bz^n}|^m \times z^{rn+sn-1}\dot{z}$.

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† *See my Mathematical Eſſays*, p. 9

From the laſt Fluent *that* of $\overline{a - bz^n}|^m \times z^{pn-1}\dot{z}$ (in which p denotes any poſitive Fraction, proper or improper) is very readily obtained: For, if the ſame (when $a - bz^n = 0$) be denoted by A; then the Fluent of $\overline{a - bz^n}|^m \times z^{pn+vn-1}\dot{z}$ will (according to the Article above quoted) be expreſſed by $\frac{p}{p+m+1} \times \frac{p+1}{p+m+2} \times \frac{p+2}{p+m+3}\ (v) \times \frac{a^v A}{b^v}$; ſuppoſing v any poſitive Integer. Therefore, by making $\overline{a-bz^n}|^m \times z^{rn+sn-1}\dot{z} = \overline{a-bz^n}|^m \times z^{pn+vn-1}\dot{z}$, or $r+s=p+v$, the correſponding Fluents muſt, alſo, be equal; that is, $\frac{p}{p+m+1} \times \frac{p+1}{p+m+2}\,(v) \times \frac{a^v A}{b^v} = \frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}\,(r) \times \frac{a^{m+p+v}}{rnb^{p+v}} \times \overline{1 - \frac{s\cdot\overline{m+1}}{t} -}$ &c. And conſequently A (the whole Fluent of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}) = \frac{p+m+1}{p} \times \frac{p+m+2}{p+1} \times \frac{p+m+3}{p+2}\,(v) \times \frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}\,(r) \times \frac{a^{m+p}}{nrb^p}$ × into the Series $1 - \frac{s\cdot\overline{m+1}}{1\cdot t} - \frac{\overline{s-1}\cdot\overline{m+2}}{2\cdot\overline{t+1}}E - \frac{\overline{s-2}\cdot\overline{m+3}}{3\cdot\overline{t+2}}F - \frac{\overline{s-3}\cdot\overline{m+4}}{4\cdot\overline{t+3}}G$ &c. where $t=r+m+1$ and $s=p+v-r$; v and r being any whole poſitive Numbers at pleaſure.

355. An Example, or two, of the Uſe of this Concluſion, may be proper.

1°. Let the *whole Fluent* of $\overline{1-x^2}|^{-\frac{1}{2}}\dot{x}$ (expreſſing the Length of $\frac{1}{4}$ of the Periphery of the Circle whoſe Radius is Unity) be demanded. In which Caſe, a being $=1$, $b=1$, $m=-\frac{1}{2}$, $n=2$, $p=\frac{1}{2}$, $t=r+\frac{1}{2}=\frac{2r+1}{2}$, and $s=v-r+\frac{1}{2}=\frac{2v-2r+1}{2}$, the Fluent ſought will, therefore, (by ſubſtituting theſe Values) be had $= \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5}\,(v) \times \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5}\,(r) \times$

$\frac{1}{2r}$ into $1 - \frac{1 \cdot \overline{2v-2r+1}}{2 \cdot \overline{2r+1}} - \frac{3 \cdot \overline{2v-2r-1}}{4 \cdot \overline{2r+3}} E - \frac{5 \cdot \overline{2v-2r-3}}{6 \cdot \overline{2r+5}} F - \frac{7 \cdot \overline{2v-2r-5}}{8 \cdot \overline{2r+7}} G$ &c. Which, by expounding v by 5 and r by 3, will become $=$ 2,16719 &c. into $1 - \frac{1 \cdot 5}{2 \cdot 7} - \frac{3 \cdot 3}{4 \cdot 9} E - \frac{5 \cdot 1}{6 \cdot 11} F + \frac{7 \cdot 1}{8 \cdot 13} G + \frac{9 \cdot 3}{10 \cdot 15} H + \frac{11 \cdot 5}{12 \cdot 17} I +$ &c. $= 1,5708$. In the bringing out of which Value, all the Terms above exhibited are requiſite: But, of the common Series, $1 + \frac{1}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} +$ &c. more than 10 times that Number of Terms would be neceſſary to anſwer with the ſame Degree of Exactneſs.

Ex. 2°. *Let the Fluxion propoſed be* $\frac{\dot{x}}{x^{\frac{1}{2}}\sqrt{d^2 - x^2}}$ (whoſe whole Fluent, when $x = d$, expreſſes the Time of Deſcent of a heavy Body in half the Arch of a Semicircle, whoſe Radius is d*.)

* Art. 207.

Here, by comparing $\overline{d^2 - x^2}\big|^{-\frac{1}{2}} \times x^{-\frac{1}{2}}\dot{x}$ with $\overline{a - bz^n}\big|^{m} \times z^{pn-1}\dot{z}$, we have $a = d^2$, $b = 1$, $n = 2$, $pn - 1 = -\frac{1}{2}$, or $p = \frac{1}{4}$; alſo $s\ (p + v - r) = v - r + \frac{1}{4}$, $t\ (r + m + 1) = r + \frac{1}{2}$: Whence, by taking r and v, each, equal to 4, the Fluent, itſelf, comes out $= \frac{3}{1} \times \frac{7}{5} \times \frac{11}{9} \times \frac{15}{13}$ into $\frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times \frac{d^{\frac{1}{2}}}{8}$ into $1 - \frac{1 \cdot 1}{4 \cdot 9} + \frac{3 \cdot 3}{8 \cdot 11} E + \frac{7 \cdot 5}{12 \cdot 13} F + \frac{11 \cdot 7}{16 \cdot 15} G +$ &c. $= \frac{11 \cdot 16 d^{\frac{1}{2}}}{13 \cdot 5}$ into $\overline{1 - \frac{1}{36} + \frac{9E}{88} + \&c.}$

$= 2{,}6215d^{\frac{1}{2}}$: Which is to $2\sqrt{2d}$, the Time of Deſcent along the vertical Diameter of the foreſaid Circle, as 2,6215 to 2,8284, or as 100 to 108, nearly.

After the ſame Manner the Fluent will be found in other Caſes: But, with regard to the aſſigning of the Values of r and v, it may be obſerved, that the Anſwer will, commonly, be brought out with the leaſt Trouble when v is taken greater by an Unit or two than r; which laſt Quantity muſt be greater or leſs, according as a greater or leſs Degree of Exactneſs is neceſſary.——From the foregoing Expreſſions, by varying the Values of v and r, a great Number of Theorems, for the Summation of Serieſes, may be deduced. But this being foreign to my preſent Purpoſe, I am not at Leiſure to purſue it here.

356. Hitherto Regard has been had to Fluxions of the Binomial-Kind: But, from thence, the Fluents of Trinomials may alſo be found; when theſe laſt can be reduced to Binomials (*by Art.* 307.) without introducing new Radical Quantities.——Beſides which Method, I ſhall, here, give another, which will anſwer where that fails, and is alſo applicable to *Multinomials.*

In order thereto, let the Fluent of $\overline{a+cz^n}^m \times z^{pn-1}\dot{z}$, be denoted by A; and let it be required to find, from thence, the Fluent of the Radical *Multinomial,* or Infinite Series, $\overline{a+cx^n+dx^{2n}+ex^{3n}+fx^{4n} \&c.}^m \times x^{pn-1}\dot{x}$.

Make $cz^n = cx^n + dx^{2n} + ex^{3n} + \&c.$ and $y = x^{pn}$; then, x^n being $= y^{\frac{1}{p}}$, if this Value be ſubſtituted for x^n, in the firſt Equation, it will become $cz^n = cy^{\frac{1}{p}} + dy^{\frac{2}{p}} + ey^{\frac{3}{p}}$ &c. Whence, by reverting the Series, (*by Art.*

Art. 275.) y (x^{pn}) is found $= z^{pn} + Rz^{pn+n} + Sz^{pn+2n} + Tz^{pn+3n} +$ &c.

Where $R = -\frac{pd}{c}$, $S = \frac{p\cdot\overline{p+3}}{2}\times\frac{d^2}{c^2} - \frac{pe}{c}$, $T = -\frac{p\cdot\overline{p+4}\cdot\overline{p+5}}{6}\times\frac{d^3}{c^3} + p\cdot\overline{p+4}\times\frac{de}{c^2} - \frac{pf}{c}$ &c.

Moreover, by taking the Fluxion of the Equation thus brought out, and dividing by pn, we have $x^{pn-1}\dot{x} = z^{pn-1}\dot{z} + \frac{p+1}{p}\times Rz^{pn+n-1}\dot{z} + \frac{p+2}{p}\times Sz^{pn+2n-1}\dot{z} + \frac{p+3}{p}\times Tz^{pn+3n-1}\dot{z} +$ &c.

Now let this Value, with that of $cx^n + dx^{2n} + ex^{3n} +$ &c. (given above) be ſubſtituted in the propoſed Fluxion, and it will become $\overline{a+cz^n}|^m \times \overline{z^{pn-1}\dot{z} + \frac{p+1}{p}\times Rz^{pn+n-1}\dot{z} + \frac{p+2}{p}\times Sz^{pn+2n-1}\dot{z} + \text{\&c.}}$

Alſo, let v denote the Place, or Diſtance, of any Term of this Series from the firſt, excluſive; then the Term itſelf, drawn into the general Multiplicator, will be expreſſed by $\overline{a+cz^n}|^m \times \frac{p+v}{p}\Delta z^{pn+vn-1}\dot{z}$ (Δ being the correſponding Coefficient R, S, T, &c.) and the Fluent thereof by

$$\frac{p+v}{p}\Delta\times\overline{a+cz^n}|^{m+1}\times z^{pn}\times\overline{\frac{z^{vn-n}}{\overline{s+1}\,.\,nc} - \frac{qaz^{vn-2n}}{\overline{s+1}\,.\,snc^2} + \frac{q\cdot\overline{q-1}\,.\,a^2z^{vn-3n}}{\overline{s+1}\,.\,s\,.\,\overline{s-1}\,.\,nc^3}\,(v)} \pm \frac{p}{t}\times\frac{p+1}{t+1}\times\frac{p+2}{t+2}\,(v)\times\frac{p+v}{p}\times\Delta\times\frac{a^vA}{c^v}$$

(*Art.* 283.)

Where, $q=p+v-1$, $s=m+q$, $t=p+m+1$, and the Sign of the laſt Term is + or −, according as v is an even or odd Number. Now, if in the Fluent thus given, v be expounded by 1, 2, 3, 4 &c. ſucceſſively, it is evident the Fluent of the whole Expreſſion will, in all Circumſtances of z, be obtained. But, if the Coefficient c be negative, ſo that $a+cz^n$ may (by increaſing z) become equal to Nothing; then, in that Circumſtance, the Fluent of the foreſaid general Term $\overline{a+cz^n}|^m \times \frac{p+v}{p} \Delta z^{pn+vn-1}\dot{z}$ (or $\overline{a-bz^n}|^m \times \frac{p+v}{p} \Delta z^{pn+vn-1}\dot{z}$, making $-c=b$) being, *barely*, $= \frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2} (v) \times \frac{p+v}{p} \times \frac{\Delta a^v A}{b^v}$ [*], it follows that the whole Fluent of the given Expreſſion, or its Equal, $\overline{a-bz^n}|^m \times \overline{z^{pn-1}\dot{z} + \frac{p+1}{p} R z^{pn+n-1}\dot{z}\ \&c.}$ will be truly repreſented by $A \times \overline{1 + \frac{\overline{p+1}\,.\,Ra}{tb} + \frac{\overline{p+1}\,.\,\overline{p+2}\,.\,Sa^2}{t\,.\,\overline{t+1}\,.\,b^2} + \frac{\overline{p+1}\,.\,\overline{p+2}\,.\,\overline{p+3}\,.\,Ta^3}{t\,.\,\overline{t+1}\,.\,\overline{t+2}\,.\,b^3}}$ &c. In which, $R=\frac{pd}{b}$, $S=\frac{p\,.\,\overline{p+3}}{2} \times \frac{d^2}{b^2} + \frac{pe}{b}$, $T=\frac{p\,.\,\overline{p+4}\,.\,\overline{p+5}}{6} \times \frac{d^3}{b^3} + \frac{p\,.\,\overline{p+4}}{1} \times \frac{de}{bb} + \frac{pf}{b}$, &c. and $A=$ the Fluent $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$, when $a-bz^n=0$.

357. Hence, if the Fluxion given be of the Trinomial Kind (then, e, f, &c. vaniſhing) the *whole Fluent* of

[*] Art. 286.

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of $\overline{a-bx^n+dx^{2n}}|^m \times x^{pn-1}\dot{x}$ (when $a-bx^n+dx^{2n}=0$) will, by ſubſtituting for *R*, *S*, *T*, &c. be $= A \times$

$$\overline{1 + \frac{p \cdot \overline{p+1}}{1 \cdot t} \times \frac{ad}{bb} + \frac{p \cdot \overline{p+1} \cdot \overline{p+2} \cdot \overline{p+3}}{1 \cdot 2 \cdot t \cdot \overline{t+1}} \times \overline{\frac{ad}{bb}}|^{2} +}$$

$$\overline{\frac{p \cdot \overline{p+1} \cdot \overline{p+2} \cdot \overline{p+3} \cdot \overline{p+4} \cdot \overline{p+5}}{1 \cdot 2 \cdot 3 \cdot t \cdot \overline{t+1} \cdot \overline{t+2}} \times \overline{\frac{ad}{bb}}|^{3}} + \text{\&c.}$$

358. If $m+1$ and p are the Halves of any odd Affirmative-Numbers, the *Fluent* of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$, when $a-bz^n=0$, will be equal to

$$\frac{1 \cdot 3 \cdot 5 \cdot 7\ (m+\frac{1}{2}) \times 1 \cdot 3 \cdot 5 \cdot 7\ (p-\frac{1}{2})}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12\ (m+p)} \times \frac{a^{m+p}G}{nb^p}{}^{*};$$

* Art. 293. and 298.

G being the Periphery of the Circle whoſe Diameter is Unity. Therefore the Fluent of $\overline{a-bx^n+dx^{2n}+ex^{3n}\ \text{\&c.}}|^m \times x^{pn-1}\dot{x}$, or its Equal, $\overline{a-bz^n}| \times \overline{z^{pn-1}\dot{z} + \frac{p+1}{p} \times Rz^{pn+n-1}\dot{z}}$ &c. is found, in this Caſe, by multiplying the Expreſſion here given, into the foregoing Series, $1 + \frac{\overline{p+1} \cdot Ra}{tb}$ + &c.

359. An Example or two will help to ſhew the Uſe of what is above delivered.

Firſt, let the *Fluent* of $\dfrac{\dot{x}}{\sqrt{a^2 - x^2 - \dfrac{x^4}{raa}}}$ (when the Diviſor becomes equal to Nothing) be required.

Then, by comparing $\overline{a^2 - x^2 - \dfrac{x^4}{raa}}|^{-\frac{1}{2}}$ with the general *Trinomial* $\overline{a - bx^n + dx^{2n}}\Big|^m \times x^{pn-1}\dot{x}$, it appears that a^2 muſt be, here, wrote in the room of a, and that n, m, p, b and d, will be interpreted by 2, $-\frac{1}{2}$, $\frac{1}{2}$, 1, and $-\frac{1}{raa}$ reſpectively: Whence we have $t\ (p+m+1) = 1$, $\frac{1.3.5\ (m+\frac{1}{2}) \times 1.3.5 (p-\frac{1}{2})}{2.4.6\ (m+p)} \times \frac{a^{m+p}G}{nb^p} = \frac{G}{2}$, and the Fluent ſought $= \frac{G}{2} \times$

$$\overline{1 - \frac{1.3}{2.2r} + \frac{1.3.5.7}{2.2.4.4r^2} - \frac{1.3.5.7.9.11}{2.2.4.4.6.6r^3} + \&c.}$$

360. The ſecond Example ſhall be, to find the Fluent expreſſing the *Apſide-Angle* in an Orbit deſcribed by means of a centripetal Force varying according to any Power of the Diſtance.

In which Caſe the given Fluxion being

$$\frac{\pm\, \mathrm{p}\dot{\mathrm{x}}}{\mathrm{x}\sqrt{\mathrm{p}^2 + \frac{2}{\mathrm{n}+1} \times \mathrm{x}^2 - \mathrm{p}^2 + \frac{2\mathrm{x}^{\mathrm{n}+3}}{\mathrm{n}+1}}}$$ (*Vid. Art.* 242.

where A is ſuppoſed the higher Apſe, and CA (and conſequently Cb) equal to Unity) we ſhall, by putting $1 - \mathrm{p}^2 = \beta$, $\frac{\mathrm{n}+3}{2} = v$, and $1 - \mathrm{x}^2 = y$, reduce it to

$$\frac{\frac{1}{2}\sqrt{1-\beta} \times \dot{y}}{\overline{1-y} \times \sqrt{\beta y + \frac{1 - vy - \overline{1-y}\big|^{v}}{1-v}}} = \frac{1}{2}\sqrt{1-\beta} \times$$

$$\frac{\overline{\beta - \frac{vy}{2} + \frac{v.\overline{v-2}}{2.3}.y^2 - \frac{v.\overline{v-2}.\overline{v-3}}{2.3.4}.y^3 + \&c.}\Big|^{-\frac{1}{2}}}{\times\, y^{-\frac{1}{2}}\dot{y} + y^{\frac{1}{2}}\dot{y} + y^{\frac{3}{2}}\dot{y} + y^{\frac{5}{2}}\dot{y} + \&c.}$$

Where the Quantity under the Radical Sign (now anſwering to the Form above preſcribed) being compared with $\overline{a - bx^n + dx^{2n} + ex^{3n} \text{ \&c.}}\,\rangle^m$, we have $m = -\frac{1}{2}$, $n = 1$, $b = \frac{v}{2}$, $\frac{d}{b} = \frac{v-2}{3}$, $\frac{e}{b} = -\frac{\overline{v-2}\,.\,\overline{v-3}}{3 \cdot 4}$ &c. Alſo the Value of p with regard to the firſt Term $(y^{-\frac{1}{2}}\dot{y})$ will be $= \frac{1}{2}$ (becauſe $pn - 1 = -\frac{1}{2}$) likewiſe its Value in the ſecond Term $(y^{\frac{1}{2}}\dot{y})$ is $= \frac{3}{2}$; in the third $= \frac{5}{2}$ &c. In the firſt of theſe Caſes we, therefore, have $t\ (m+p+1) = 1$, $R\ \left(p \times \frac{d}{b}\right) = \frac{v-2}{6}$, $S = \frac{\overline{v-2}\,.\,\overline{4v-5}}{72}$, $T = \frac{\overline{v-2}\,.\,\overline{16v^2-37v+22}}{16 \times 45}$. Whence it follows, that the Fluent of the firſt Term $\left(\overline{\beta - \frac{vy}{2} + \frac{v\,.\,\overline{v-2}}{2\,.\,3}\,.\,y^2 \text{ \&c.}}\,\right|^{-\frac{1}{2}} \times y^{-\frac{1}{2}}\dot{y}\Big)$ when the Quantity under the Radical Sign becomes equal to Nothing (or the Body arrives at its lower Apſe) will be truly expreſſed by $\frac{G}{\sqrt{\frac{1}{2}v}}$ into $1 + \frac{v-2}{2v}\,.\,\beta + \frac{5\,.\,\overline{v-2}\,.\,\overline{4v-5}}{48v^2}\,.\,\beta^2 + \frac{7\,.\,\overline{v-2}\,.\,\overline{16v^2-37v+22}}{6 \times 48v^3}\,.\,\beta^3$ + &c.

In the ſame Manner it will appear, that the Fluent of the ſecond Term, in that Circumſtance, is $= \frac{G}{\sqrt{\frac{1}{2}v}} \times \overline{\frac{1}{v}\,.\,\beta + \frac{5\,.\,\overline{v-2}}{4v^2}\,.\,\beta^2 + \frac{35\,.\,\overline{v-2}\,.\,\overline{2v-3}}{48v^3}\,.\,\beta^3}$

&c. that of the Third $= \frac{G}{\sqrt{\frac{1}{2}v}} \times \overline{\frac{3}{2v^2}.\beta^2 + \frac{35.\overline{v-2}}{12v^3}.\beta^3}$ &c. that of the Fourth $= \frac{5}{2v^3}.\beta^3$ &c. &c.

Whence, the Fluent of the whole Series, by collecting theſe ſeveral Values together, will come out =

$$\frac{G}{\sqrt{\frac{1}{2}v}} \times \overline{1 + \tfrac{1}{2}\beta + \frac{20v^2 - 5v + 2}{48v^2}.\beta^2 + \frac{112v^3 - 63v^2 - 42v - 8}{6 \times 48v^3}.\beta^3 +}$$

&c. Which, drawn into $\frac{1}{2} \times \overline{1 - \frac{1}{2}\beta - \frac{1}{6}\beta^2 - \frac{1}{16}\beta^3 -}$ &c. (the Value of the general Multiplicator $\frac{1}{2}\sqrt{\overline{1-\beta}}$) gives

$$\frac{G}{\sqrt{2v}} \times \overline{1 \quad * \quad + \frac{\overline{v-2}.\overline{2v-1}}{48}.\frac{\beta^2}{v^2} + \frac{\overline{v-2}.\overline{2v-1}.\overline{2v-1}}{72} \times \frac{\beta^3}{v^3}}$$

&c. for the true Meaſure of the Angle required, in Parts of the Radius, or Unity: From whence, by writing 180 inſtead of G, we ſhall have the ſame in Degrees: Which, laſt of all, by reſtoring n, becomes

$$\frac{180^\circ}{\sqrt{n+3}} \times \overline{1 \quad * \quad + \frac{\overline{n-1}.\overline{n+2}}{24} \times \overline{\frac{\beta}{n+3}}\Big|^2 + \frac{\overline{n-1}.\overline{n+2}.\overline{n+2}}{18} \times \overline{\frac{\beta}{n+3}}\Big|^3}$$

&c.

Where n is the Exponent of the Law of the Force, whereby the Orbit is deſcribed; and β, the Defect of the Square of the Meaſure of the Celerity, at the higher *Apſe*, below That which the Body ought to have to revolve in a Circle, this laſt being denoted by Unity.

The ſame Concluſion may be otherwiſe derived, by bringing $1-y$, in the transformed Fluxion, under the *Vinculum*; but this Way of going to work, though we have but one Series to manage, will prove rather more troubleſome than the foregoing.

It will appear from the two preceding Examples, eſpecially the firſt of them, that this laſt Method of finding Fluents is, chiefly, uſeful when all the Terms of the given Expreſſion, after the two firſt, in reſpect of theſe, are but ſmall. Which is a Circumſtance that frequently occurs in the Reſolution of phyſical Problems; ſuch as determining the Effect of the Atmoſphere's Reſiſtance upon the Vibration of Pendulums; and the *Inequalities* of the Planets ariſing from their Action on each other.——In ſhort, wherever the Fluent, or the Quantity it expreſſes, would belong to the Circle, or ſome other of the Conic-Sections, were it not for the Interpoſition of ſome ſmall perturbating Force (whereby new Terms, ſmall in Compariſon of the two firſt, are introduced) the ſaid Method will be found of very great Service.

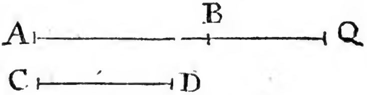
SECTION VIII.

The Uſe of Fluxions in determining the Motion of Bodies in reſiſting Mediums.

PROB. I.

361. *Suppoſing that a Body, let go from a given Point* A, *with a given Celerity, in a Right-line* AQ, *is reſiſted by a Medium (or any Force) acting according to a given Power of the Velocity: To determine the Velocity, and alſo the Space run over, at the End of a given Time.*

LET the given Celerity at A (meaſur'd by the Space which would be uniformly deſcribed in any propoſed Time r) be put $= c$, and that at any other Point B, $= v$; moreover put AB $= x$, and the Time of its Deſcription $= z$; and let the Reſiſtance, or Force, acting upon the Body at A, be ſuch, that, if the ſame was to be uniformly continued, the Body would have all its Motion deſtroyed thereby, in the Time wherein it might move, uniformly, over a given Diſtance d (CD) with its firſt Velocity c: Which Time, let be denoted, by t.



Then, ſince the whole Celerity c would be deſtroy'd in the Time t, that Part of it which would be uniformly taken away in the Time r, above propoſed, will be truly repreſented by $\frac{r}{t} \times c$; or by $\frac{cc}{d}$; which is equal to it, becauſe the Spaces (c and d) deſcribed with the ſame Ce-

Celerity are always as the Times (*r* and *t*) of their Deſcription; and therefore $\frac{r}{t}=\frac{c}{d}$.

Hence, the Reſiſtance at B being to that at A (*by Hypotheſis*) as v^n to c^n, it follows that the Velocity which might be deſtroyed in the given Time *r*, by a Force equal to the Reſiſtance at B, will be expreſſed by $\frac{cc}{d}\times\frac{v^n}{c^n}$, or its Equal $\frac{v^n}{dc^{n-2}}$: Which Expreſſion is, therefore, the true Meaſure of the Force of the ſaid Reſiſtance.

Now, it appears, from *Art.* 218. that, if the Force with which the Body is acted on (or the Velocity it would generate in the given Time *r*) be repreſented by *F*, the Relation of the Meaſures of the Velocity and Space gone over, will be expreſſed by the Equation $\pm v\dot{v}=F\dot{x}$: From whence, by writing $\frac{v^n}{dc^{n-2}}$ inſtead of *F*, we have $-v\dot{v}=\frac{v^n\dot{x}}{dc^{n-2}}$ (the Sign of $v\dot{v}$ being negative, becauſe v decreaſes while x increaſes *.) * Art. 5.
From this Equation, we get $\dot{x}=-dc^{n-2}v^{1-n}\dot{v}$; whoſe Fluent is $x=-\frac{dc^{n-2}\times v^{2-n}}{2-n}+$; which, corrected (by taking $x=0$, and $v=c$) becomes $x=$

$$\frac{-dc^{n-2}\times v^{2-n}+d}{2-n}=\frac{d}{n-2}\times\overline{\left.\frac{c}{v}\right|^{n-2}-1}.$$

Moreover, ſince the Time ($\dot{z}$) is to the Time *r*, as the Diſtance $\dot{x}$ to the Diſtance *v*, we alſo have $\dot{z}$ $\left(=\frac{r\dot{x}}{v}\right)=-rdc^{n-2}v^{-n}\dot{v}$; and conſequently $z=$

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$\frac{rd}{\overline{n-1}\times c}\times\overline{\frac{c}{v}\Big)}^{n-1}-1=\frac{t}{n-1}\times\overline{\frac{c}{v}\Big)}^{n-1}-1$ (by writing t for its Equal $\frac{rd}{c}\Big)$: From which Equation we get $\frac{c}{v}=\overline{1+\overline{n-1}\times\frac{z}{t}\Big)}^{\frac{1}{n-1}}$: Likewiſe, from the preceding Equation, we get $\frac{c}{v}=$ $\overline{1+\overline{n-2}\times\frac{x}{d}\Big|}^{\frac{1}{n-2}}$: Which two equal Values being compared together, there, at length, reſults $x=$ $\frac{d}{n-2}$ into $\overline{1+\overline{n-1}\times\frac{z}{t}\Big|}^{\frac{n-2}{n-1}}-1$, for the required Relation of x and z. *Q. E. I.*

COROLLARY.

362. If $n=2$, or, the Reſiſtance be in the Duplicate Ratio of the Velocity, the Equation exhibiting the Relation of z and v, will be $\frac{c}{v}=1+\frac{z}{t}$, or $v=$ $\frac{c}{1+\frac{z}{t}}$: But the other Equation (the Fluent failing) becomes impracticable. Here x, the Fluent of $-\frac{d\dot{v}}{v}$, will be explicable by $d\times$ *hyp. Log.* $\frac{c}{v}$ *, or by $d\times$ *hyp. Log.* $1+\frac{z}{t}$; becauſe $v=\frac{c}{1+\frac{z}{t}}$.

* Art. 126.

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In the like Manner, when $n=1$, or the Reſiſtance is as the Velocity, the Relation of v, x and z, will be exhibited by the Equations $v = c \times \frac{\overline{d-x}}{d}$, and $z = t \times$ *hyp. Log.* $\frac{c}{v} = t \times$ *hyp. Log.* $\frac{d}{d-x}$. Which Caſe, and that above, are the only two wherein the general Solution fails.

PROB. II.

363. *If a Body, let go from a given Point* A *with a given Celerity, in a vertical Line* CAQ, *is acted on by an uniform Gravity, and alſo by a Medium, reſiſting according to any given Power of the Velocity; 'tis propoſed to determine the Relation of the Times, the Velocities, and the Spaces gone over.*

Let the Notation in the preceding Problem be retained; and let the Force of Gravity, in the given Medium (meaſured by the Velocity it might generate in the propoſed Time r *) be repreſented by b. Then, * Art. 361.
this Value being added to, or ſubſtracted from $\left(\frac{v^n}{dc^{n-2}}\right)$ the Meaſure of the Reſiſtance †, according as the Body is in its Aſcent, or † Art. 361.
Deſcent, we thence get $\frac{v^n}{dc^{n-2}} \pm b$ for the whole Force (F) whereby the Motion, at B, is affected: Whence (*by Art.* 218) $\dot{x}\ \left(= \frac{-v\dot{v}}{F}\right) = \frac{-dc^{n-2}v\dot{v}}{v^n \pm bdc^{n-2}}$; and $\dot{z}\ \left(= \frac{r\dot{x}}{v} *\right) = \frac{-rdc^{n-2}\dot{v}}{v^n \pm bdc^{n-2}}$: Whoſe Fluents * Art. 361.

may

may be had, by the Means of circular Arcs, and Logarithms, *from Art.* 331. *Q. E. I.*

Corollary I.

364. It appears that the Force $\left(\frac{v^n}{dc^{n-2}}\right)$ of the Reſiſtance is to (b) *that* of Gravity, in the given Medium, as v^n to bdc^{n-2}: Therefore, if this Ratio be expounded by that of v^n to a^n, or a^n be put $= bdc^{n-2}$, it follows that a will expreſs the Celerity with which the Reſiſtance would be equal to the Gravity (ſince, when $v = a$, the ſaid Ratio becomes that of Equality.) Hence, alſo, by ſubſtituting $\frac{a^n}{b}$ for its Equal dc^{n-2}, we get

$$\dot{x} = \frac{-a^n v\dot{v}}{b \times \overline{v^n \pm a^n}}, \text{ and } \dot{z} = \frac{-ra^n\dot{v}}{b \times \overline{v^n \pm a^n}}.$$

Corollary II.

365. If the Reſiſtance be in the Duplicate Ratio of the Celerity, our two laſt Equations will become $\dot{x} = \frac{-a^2 v\dot{v}}{b \times \overline{vv \pm aa}}$, and $\dot{z} = \frac{-ra^2\dot{v}}{b \times \overline{vv \pm aa}}$: From the former whereof we get $x = -\frac{a^2}{2b} \times hyp.\ Log.\ \frac{vv \pm aa}{cc \pm aa}$*

* Art. 126.

$= \frac{a^2}{2b} \times hyp.\ Log.\ \frac{cc \pm aa}{vv \pm aa} = \frac{d}{2} \times hyp.\ Log.\ \frac{cc \pm bd}{vv \pm bd}$ (becauſe, here, $a^2 = bd$.) From whence, when $v = 0$, (ſuppoſing the Body to aſcend) there comes out $x = \frac{d}{2} \times hyp.\ Log.\ \overline{1 + \frac{cc}{aa}}$, for the Height (AQ) of the whole Aſcent. But, if c be taken $= 0$, or the Body be ſuppoſed to deſcend from Reſt, we ſhall then have $-\frac{d}{2}\times$ *hyp. Log.* $\overline{1-\frac{vv}{aa}}=$ the Diſtance *AB* deſcended. Whence, if *N* be put for the Number whoſe Hyperbolical Logarithm is $\frac{2x}{d}$, it follows, (becauſe, Log. $1-\frac{vv}{aa}=-\frac{2x}{d}=-$ Log. N) that $1-\frac{vv}{aa}=\frac{1}{N}$, and conſequently $v=a\sqrt{\frac{N-1}{N}}$. From which, the Diſtance *AB* being given, the Velocity acquired in the Fall will be determined. But, if the Body, firſt, aſcends from a given Point *A*, with a given Celerity *c*, and the Celerity, acquired in falling, when it arrives, again, at that Point, be required; the ſame may be exhibited in a more commodious Form, independent of Logarithms, and will be equal to $\frac{c}{\sqrt{1+\frac{cc}{aa}}}$; becauſe *N*, in this Caſe, is found above to be $=1+\frac{cc}{aa}$. Furthermore, with regard to the Time (z), we have already found that $\dot{z}$ is $=\frac{-ra^2\dot{v}}{b\times\overline{vv+aa}}$, or $=\frac{-ra^2\dot{v}}{b\times\overline{vv-aa}}$ $\left(=\frac{ra^2\dot{v}}{b\times\overline{aa-vv}}\right)$ according as the Motion of the Body is from, or towards the Center of Force. Therefore the Time itſelf, in the former Caſe, will be $=\frac{ra}{b}$ drawn into the Difference of the two circular Arcs whoſe Tangents are $\frac{c}{a}$ and $\frac{v}{a}$, and whereof the common Radius is Unity *: Whence it follows that the

Time

* Art 142.

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Time of the whole Aſcent will be denoted by $\frac{ra}{b}$ multiply'd into the former of the ſaid Arcs.

But, in the other Caſe, the Fluent, exhibiting the Time of Deſcent, is not explicable by the Arcs of a Circle, but by the Difference of the hyperbolical Logarithms of $\frac{a+v}{a-v}$ and $\frac{a+c}{a-c}$ drawn into $\frac{ra}{2b}$ *. Therefore, when $c = 0$, or the Body falls from Reſt, the Time z will be barely $= \frac{ra}{2b} \times hyp.\ Log.\ \frac{a+v}{a-v} = \frac{ra}{b} \times hyp.\ Log.\ N^{\frac{1}{2}} + \overline{N-1}|^{\frac{1}{2}}$ (by ſubſtituting the Value of v found above, and ordering the Logarithm as in *Art.* 303.) This Equation, in the forementioned Circumſtance, where $N = 1 + \frac{cc}{aa}$, and $v = \frac{c}{\sqrt{1+\frac{cc}{aa}}}$, becomes $z = \frac{ra}{b} \times hyp.\ Log.\ \overline{\sqrt{1+\frac{cc}{aa}} + \frac{c}{a}}$.

* Art. 126.

SCHOLIUM.

366. If, according to Sir *Iſaac Newton*, we ſuppoſe the Reſiſtance of the Air, to Bodies moving in it, to be in the Duplicate Ratio of the Celerities *; and that

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* *That the Reſiſtance is as the Square of the Celerity, the Learner may, in ſome meaſure, conceive, by conſidering that the ſame Body, with a double Velocity, not only puts twice the Number of reſiſting Particles in Motion, in the ſame time, but alſo acts upon each with a double Force; and therefore muſt ſuffer a four-fold Reſiſtance, or a Reſiſtance proportional to the Square of the Velocity. This would be ſtrictly true, were it not that the Particles ſo put in Motion impel others lying before them, and thereby prevent, as it were, the Action of the Body. What Deviation from the foregoing Law may hence ariſe, is not eaſy to determine. This, however, ſeems plain, that the Reſiſtance at the Beginning of any very ſwift Motion (till the Air in the Way of the Body comes duly to participate of that Motion) will be greater than That ſuſtained by another equal Body, moving with the ſame Celerity, that has been in Motion ſome time.*

a Ball, in the Time it might move, uniformly, over a Space *(d)* which is to $\frac{8}{3}$ of its Diameter as the Denſity of the Ball to that of the Medium, would have all its Motion taken away by a Force equal to that of the Reſiſtance, uniformly continued: Then, from theſe *Data*, apply'd to the Theorems in the preceding Article, we ſhall be able to determine the Velocities, and the Times of the perpendicular Aſcent and Deſcent of Bodies near the Earth's Surface; allowing for the Reſiſtance of the Atmoſphere.

Thus, for Inſtance, let a Canon Ball, of four Inches Diameter (whereof the Denſity, or ſpecific Gravity, is to that of Air as 6000 to 1, nearly) be ſuppoſed to be projected, perpendicular to the Horizon, with a Velocity ſufficient to cauſe it to aſcend to the Height of half a Mile, or 2640 Feet, *in vacuo*; which Velocity *(by Art.* 203.) will be found to anſwer to the Rate of about 412 Feet *per Second:* Then, according to the Proportion juſt now mention'd, it will be as 1 : 6000 :: $\frac{8}{3}\times 4$: 64000 Inches, or 5333 Feet; which is the Value of *d* in this Caſe. Therefore, if the Time *r*, in the preceding Article (which may be aſſumed at pleaſure) be here interpreted by *one Second*, the correſpondent Values of *d*, *c* and *b* will be expounded by 5333 F. 412 F. and $32\frac{1}{12}$ F. * reſpectively. Which Values being ſubſtituted in the ſeveral Equations in the laſt Article, we ſhall get

* Art. 202.

1°. $a\ (=\sqrt{bd}) = 414$ F. the Velocity, *per Second*, wherewith the Reſiſtance would be equal to the Gravity, or Weight, of the Ball.

2°. $\frac{d}{2}\times$ *hyp. Log.* $1+\frac{cc}{aa} = 1835$ Feet, the whole Height of the Aſcent.

3°. $\frac{ra}{b}\times$ *Arch.* whoſe Tang. is $\frac{c}{a} = 10{,}08$ *Seconds*, the whole Time of the Aſcent (which is leſs than the Time, *in vacuo*, by 2,73.)

4°. $v\ (= \dfrac{c}{\sqrt{1+\frac{cc}{aa}}}) = \overset{F}{292}$, the Velocity, *per Second*, acquired in the Deſcent.

5°. Laſtly, $\frac{ra}{b} \times$ *hyp. Log.* $\sqrt{1+\frac{cc}{aa}} + \frac{c}{a} =$ 11,30 Seconds, the Time of the Deſcent.

Note, In this Example the Meaſure of the abſolute Gravity of the Body, *in vacuo*, is taken, inſtead of its Gravity in Air (the Difference, *there*, being too inconſiderable to be regarded.) But, in Caſes where the ſpecific Gravity of the Medium bears a ſenſible Proportion to that of the Body, the Force of Gravity *(b)* muſt be expounded by $32\frac{1}{12} \times \frac{B-M}{B}$ (inſtead of $32\frac{1}{12}$) where *B* is to *M* as the ſpecific Gravity of the Body to that of the Medium.

PROB. III.

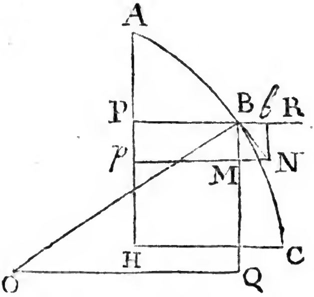
367. *To determine the Reſiſtance, by means whereof a Body, gravitating uniformly in the Direction of parallel Lines, may deſcribe a given Curve.*

Let ABC be the given Curve, and let BQ, parallel to the Axis (or any given Line) AH, be the Direction of Gravitation at any Point B: Make PBR perpendicular to AH and BQ; and let AP$=x$, PB$=y$, AB$=z$, BM (N*b*) $=\dot{x}$, MN (B*b*) $=\dot{y}$, BN $=\dot{z}$, and the Velocity of the Body at B in the Direction PBR $=v$. Then, the Decreaſe of Velocity in the ſaid Direction, which is wholly owing to the Reſiſtance *, being repreſented by $-\dot{v}$, it follows that the correſponding Decreaſe of Motion in the Direction BN, ariſing from the ſame Cauſe, will be expreſſed by $\frac{\dot{z}}{\dot{y}} \times -\dot{v} = -\frac{\dot{v}\dot{z}}{\dot{y}}$; and, that in the Direction BM, by $-\frac{\dot{v}\dot{x}}{\dot{y}}$. But, the

* Art. 209.

of

Celerity in this laſt Direction being, every where, repreſented by $v \times \frac{\dot{x}}{\dot{y}}$, its Fluxion $\frac{v\ddot{x} + \dot{v}\dot{x}}{\dot{y}}$ will be the



whole Alteration of Motion in the ſaid Direction, ariſing from the Reſiſtance and the Force of Gravity, conjunctly: From which deducting the Part owing to the Reſiſtance, found above to be $\frac{\dot{v}\dot{x}}{\dot{y}}$, the Remainder $\frac{v\ddot{x}}{\dot{y}}$ will be the Effect of the Gravity. Which being to $\left(-\frac{\dot{v}\dot{z}}{\dot{y}}\right)$ the Effect of the abſolute Reſiſtance in the Direction BN, as 1 to $-\frac{\dot{v}\dot{z}}{v\ddot{x}}$, the Force of Gravity, muſt therefore be to that of the required Reſiſtance, in the ſame Ratio of 1 to $-\frac{\dot{v}\dot{z}}{v\ddot{x}}$.

Moreover, the Force of Gravity, meaſur'd by the Velocity it would generate in a given Part of Time (1), being denoted by Unity, the Velocity generated thereby, in the Time $\left(\frac{\dot{y}}{v}\right)$ of deſcribing B*b*, with the Celerity v, will likewiſe be truly expreſſed by, $\frac{\dot{y}}{v}$, the Meaſure of the ſaid Time: Which being put = to $\left(\frac{v\ddot{x}}{\dot{y}}\right)$ the Value of the ſame Quantity, given above, we thence have $v^2 = \frac{\dot{y}\dot{y}}{\ddot{x}}$: From whence, not only the Velocity, but the Reſiſtance will be found. But, if you would have the Reſiſtance expreſſed independent of v; then let the Fluxion $\left(2v\dot{v} = -\frac{\dot{y}^2\dddot{x}}{\ddot{x}\ddot{x}}\right)$ of the laſt Equation be divided by the Fluent, which will give $\frac{\dot{v}}{v} = -\frac{\frac{1}{2}\dddot{x}}{\ddot{x}}$: And then, by ſubſtituting this Value in $-\frac{\dot{v}\dot{z}}{v\ddot{x}}$, you will get $\frac{\dot{z}\dddot{x}}{2\ddot{x}\ddot{x}}$, for the true Force of the Reſiſtance, *that* of Gravity (or the Weight of the Body) being expounded by Unity.

The ſame otherwiſe.

Let BO be the Radius of Curvature at B, and let OQ be parallel to PB, meeting BM, produced, in Q: Then, if the abſolute Gravity, acting in the Direction BQ, be denoted by Unity, its Force in the Direction BO, whereby the Body is retained in the Curve, will be repreſented by $\frac{BQ}{BO}$. Therefore, ſince the Velocities in Circles are known to be in the Subduplicate Ratio of the Radii and of the Forces conjunctly *, the Velocity at B will be rightly expreſſed by $\sqrt{BO \times \frac{BQ}{BO}}$, or its Equal $\sqrt{BQ}$. (For the Curve at, and indefinitely near, B may be taken as an Arch of a Circle whoſe Radius is BO: And it is evident that the Reſiſtance has nothing to do in forcing the Body from the Tangent,

* Art. 212.

Tangent, but only ſerves to retard its Motion ſo, that it may, every where, bear a due Proportion to the given Force of Gravity acting in the Direction BO.) Hence, putting BQ $= s$, the Increaſe of the Celerity in the Time $\left(\frac{\dot{z}}{\sqrt{s}}\right)$ of deſcribing BN, will be expreſſed by the Fluxion of $\sqrt{s}$, or $\frac{\dot{s}}{2\sqrt{s}}$. Moreover, the Celerity that might be generated by Gravity in the ſaid Time $\frac{\dot{z}}{\sqrt{s}}$ being meaſured thereby, the Increaſe, in BN, ariſing from the ſame Cauſe, will therefore be $= \frac{\dot{z}}{\sqrt{s}} \times \frac{\dot{x}}{\dot{z}} = \frac{\dot{x}}{\sqrt{s}}$: Which, being taken from $\left(\frac{\dot{s}}{2\sqrt{s}}\right)$ the whole Increaſe, found above, the Remainder, $\frac{\dot{s} - 2\dot{x}}{2\sqrt{s}}$, will be the Effect of the Reſiſtance: Which is to the Effect, $\frac{\dot{z}}{\sqrt{s}}$, of the abſolute Gravity as $\frac{\dot{s} - 2\dot{x}}{2\dot{z}}$ to 1. Therefore the Reſiſtance is to the Gravity (or Weight of the Body) as $\frac{2\dot{x} - \dot{s}}{2\dot{z}}$ to Unity: Where the Signs are changed, becauſe the two Forces act in contrary Directions.

Becauſe BO $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$ *, therefore s $\left(\text{BO} \times \frac{\dot{y}}{\dot{z}}\right) =$ * Art. 68.
$\frac{\dot{z}^2}{\ddot{x}} = \frac{\dot{y}^2 + \dot{x}^2}{\ddot{x}}$ (= the Square of the Celerity) whence $\dot{s} = \frac{2\dot{x}\ddot{x}\ddot{x} - \overline{\dot{y}^2 + \dot{x}^2} \times \dddot{x}}{\ddot{x}\ddot{x}}$, and conſequently the Re-

ſiſtance $\frac{2\dot{x}-\dot{s}}{2\dot{z}}=\frac{\overline{\dot{y}^2+\dot{x}^2}\times\dddot{x}}{2\dot{z}\dot{x}\ddot{x}}=\frac{\dot{z}\dddot{x}}{2\dot{x}\ddot{x}}$, *the very ſame as before.*

COROLLARY.

368. If the Reſiſtance be ſuppoſed as any given Power of the Velocity drawn into (D) the Denſity of the Medium; then, from hence, the Denſity of the Medium, at every Point of the Curve, may be determined: For, the abſolute Celerity at B being repreſented by $\frac{v\dot{z}}{\dot{y}}$, the Reſiſtance at that Point will, according to the ſaid Hypotheſis, be as $\overline{\frac{v\dot{z}}{\dot{y}}}\Big|^{n}\times D$; and therefore the Velocity that would be deſtroyed thereby, in the Time $\left(\frac{\dot{y}}{v}\right)$ of deſcribing BN, as $\overline{\frac{v\dot{z}}{\dot{y}}}\Big|^{n}\times\frac{D\dot{y}}{v}$: Which being put $=\left(-\frac{\dot{v}\dot{z}}{\dot{y}}\right)$ the Effect of the ſame Reſiſtance, found above, we thence get $D=\frac{-\dot{v}\dot{y}^{n-2}}{\overline{v\dot{z}}|^{n-1}}$: Which, by ſubſtituting for v and $\dot{v}$, becomes $D=\frac{\dddot{x}}{2\dot{z}^{n-1}\times\ddot{x}^{2-\frac{1}{2}n}}$.

In this Corollary, and what, elſewhere, relates to unequal Denſities, the Gravity of the Body in the Medium is ſuppoſed to continue, every where, the ſame, or, that the Attraction increaſes with the Denſity, ſo that the Difference between the ſpecific Gravities of the Body and Medium may, at every Point, be a conſtant Quantity.

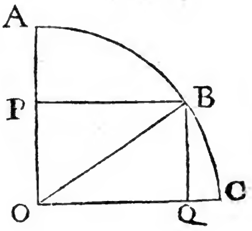
EXAMPLE I.

369. *Let the propoſed Curve* ABC *be the common Parabola:*

Then, x being here $= \frac{y^2}{a}$, we have $\dot{x} = \frac{2y\dot{y}}{a}$, $\ddot{x} = \frac{2\dot{y}\dot{y}}{a}$ and $\dddot{x} = 0$; and therefore $\frac{\dot{z}\dddot{x}}{2\dot{x}\ddot{x}}$* is alſo $= 0$: * Art. 367. Whence it appears that a Body, to deſcribe this Curve, muſt move in Spaces intirely void of Reſiſtance.

EXAMPLE II.

370. *Let the Curve* ABC *be taken as a Quadrant of a Circle, whoſe Radius* BO *is* $= a$.



In this Caſe we have s (BQ) * $= a - x$ ($=$AO $-$AP) whence $\dot{s} = -\dot{x}$, and therefore $\frac{2\dot{x} - \dot{s}}{2\dot{z}} = \frac{3\dot{x}}{2\dot{z}} = \frac{3\text{PB}}{2\text{AO}}$\*. From which it is evident, that the Velocity is, every where, as $\sqrt{\text{BQ}}$, and the Reſiſtance to the Gravity (or Weight of the Body) as 3PB to 2OB.

* Art. 367.

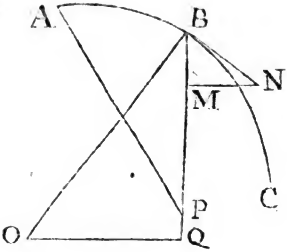
* Art. 142.

PROB. IV.

371. *The Centripetal Force (F) being given; to find the Reſiſtance and Velocity whereby a Body may deſcribe a given Spiral (or any other, poſſible, Curve) about the Center of Force.*

Let P be the Center of Force, and BO the Radius of Curvature at any Point B in the propoſed Curve.

and



and let OQ be perpendicular to BPQ; alſo let BP $= y$, BQ $= s$, AB $= z$, BM $= -\dot{y}$ *, and BN $= \dot{z}$. Then, it is evident from *Art.* 367. that the Velocity at B will be expreſſed by $\sqrt{\text{BO} \times \frac{\text{BQ}}{\text{BO}} \times F}$; or, its Equal, $\sqrt{sF}$: And therefore its Increaſe in the Time $\left(\frac{\dot{z}}{\sqrt{sF}}\right)$ of deſcribing BN will be $\frac{s\dot{F} + F\dot{s}}{2\sqrt{sF}}$: From which, deducting $(F \times \frac{\dot{z}}{\sqrt{sF}} \times \frac{-\dot{y}}{\dot{z}})$ the Effect of the centripetal Force, in the ſame Time and Direction, the Remainder, $\frac{s\dot{F} + F\dot{s} + 2F\dot{y}}{2\sqrt{sF}}$, is the Effect of the Reſiſtance. Therefore the Reſiſtance is to the centripetal Force as $\frac{s\dot{F} + F\dot{s} + 2F\dot{y}}{2\sqrt{sF}}$ to $\frac{F\dot{z}}{\sqrt{sF}}$, or as $\frac{s\dot{F} + F\dot{s} + 2F\dot{y}}{2F\dot{z}}$ to Unity. *Q. E. I.*

* Art. 5.

EXAMPLE.

372. Let the Meaſure (F) of the centripetal Force be expounded by any Power y^n of the Diſtance; and let the Curve be the logarithmic Spiral; putting the Co-ſine of the given Angle PBN † (to the Radius r) $= c$. Then, s being here $= y$ ‡, and $\dot{F} = ny^{n-1}\dot{y}$, we

† Art. 61.

‡ Art. 74.

we have $\frac{s\dot{F} + F\dot{s} + 2F\dot{y}}{2F\dot{z}} = \frac{ny^n\dot{y} + y^n\dot{y} + 2y^n\dot{y}}{2y^n\dot{z}} = \frac{n+3}{2} \times \frac{\dot{y}}{\dot{z}} = \frac{n+3}{2} \times \frac{c}{r}$.

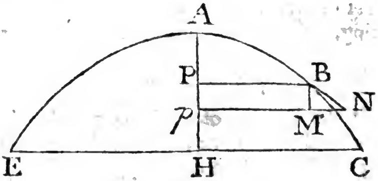
Hence it appears that the Velocity muſt be, every where, as $y^{\frac{n+1}{2}}$; and the Reſiſtance, to the centripetal Force, as $\frac{n+3}{2} \times \frac{c}{r}$ to Unity. But, when $n = -3$, $\frac{n+3}{2} \times \frac{c}{r}$ becomes $= 0$; therefore the Body, in this Caſe, muſt move in Spaces intirely void of Reſiſtance; agreeable to *Art.* 233. And, if $n+3$ be negative, an accelerating, inſtead of a reſiſting Force, will be required.

SCHOLIUM.

373. If the Denſity of a Medium, wherein a Body moves, be either uniform, or varies according to a given Law, the Nature of the Curve, or Trajectory, may be determined from what is delivered in the preceding Pages.

Thus, for Example, let the Denſity be ſuppoſed every where the ſame, and the Reſiſtance as the Square of the Celerity; then, from *Art.* 368. we have $\frac{\dddot{x}}{\dot{z}\ddot{x}} = D$; which, in order to exterminate $\dot{z}$, may be transformed to $\dddot{x}\dddot{x} = \overline{\dot{y}\dot{y} + \dot{x}\dot{x}} \times D^2\ddot{x}\ddot{x}$: Where, D being a conſtant Quantity (depending upon the given Denſity of the Medium) the Value of x will be found, as is taught in *Sect.* 2. *Art.* 268. 271. and comes out $= \frac{y^2}{p} + \frac{Dy^3}{3p} + \frac{D^2y^4}{12p}$ &c. In which p is put to denote the Parameter of the Curve at the Vertex, or higheſt Point A, (to be determin'd from the Force of Gravity and the given Velocity of the Body at that Point.) This Solution anſwers near enough when the Reſiſtance is but ſmall in Proportion to the Gravity; in other Circumſtances, the Series not converging, it becomes uſeleſs: For which Reaſon, and becauſe the Caſe above ſpecified is That ſuppoſed to obtain, in reſpect to the Air near the Earth's Surface, and its Reſiſtance to Bodies moving therein, I ſhall ſhew, by a different Method, how the Nature of the Curve may be inveſtigated.

In order thereto, let the Celerity at the higheſt Point, A, above the Plane of the Horizon EC, be denoted by c; and let a be the Celerity with which the Reſiſtance is equal to the Gravity *(vid. Art.* 365. *and* 366.)



Moreover, let d be put for the Diſtance over which the Ball might uniformly move in the Time that the Medium would deſtroy all its Motion, was the Reſiſtance to continue the ſame, all along, as at the firſt Inſtant (Which Diſtance, according to Sir *Iſaac Newton*, is, always, in Proportion to $\frac{8}{3}$ of the Ball's Diameter, as the Denſity of the Ball to that of the Medium.)

Then it will be, as $d : \dot{z}$ (BN) :: $\frac{v\dot{z}^2}{\dot{y}}$, the abſolute Celerity at B, to $\left(\frac{v\dot{z}^2}{d\dot{y}}\right)$ the Part thereof that would be uniformly deſtroyed by the Reſiſtance in the Time of deſcribing BN, with the Velocity at B: Which Value being alſo expreſſed by $\frac{-\dot{v}\dot{z}}{\dot{y}}$ (*vid. Art.* 367.) we therefore have $\frac{v\dot{z}^2}{d\dot{y}} = -\frac{\dot{v}\dot{z}}{\dot{y}}$; whence $\frac{\dot{z}}{d} = \frac{-\dot{v}}{v}$, and conſequently, by taking the Fluent, $\frac{z}{d} = -$ hyp. Log. v; which corrected (by putting $z=0$, and $v=c$) gives $\frac{z}{d}$ (=hyp. Log. c—hyp. Log. v) =hyp. Log. $\frac{c}{v}$.

Furthermore, ſince (by Hypotheſis) the Reſiſtance with the Celerity $\frac{v\dot{z}}{\dot{y}}$ (at B) is to the Force of Gravity, or the Reſiſtance with the Celerity a, as $\frac{vv\dot{z}\dot{z}}{\dot{y}\dot{y}}$ to a^2; and it appears, from the aforeſaid Article, that the ſame Ratio is alſo univerſally expreſſed by *that* of $\frac{-\dot{v}\dot{z}}{v\ddot{x}}$ to 1, it follows, from the Equality of theſe Ratios, that $\frac{\dot{z}\ddot{x}}{\dot{y}\dot{y}}$ is $= -\frac{a^2\dot{v}}{v^3}$. But, in order to the Reſolution of the Equation thus given, let the Tangent of the Angle PBA (or N) which the Ordinate, PB, makes with the Curve (ſuppoſing Radius Unity) be, every where, repreſented by w: Then, becauſe $\dot{x}=w\dot{y}$, $\dot{z}\ (\sqrt{\dot{y}^2+\dot{x}^2}) = \dot{y}\sqrt{1+w^2}$, and $\ddot{x} = \dot{w}\dot{y}$ ($\dot{y}$ being conſtant) we ſhall, by ſubſtituting theſe Values in the foreſaid Equation, get $-\frac{a^2\dot{v}}{v^3} = \dot{w}\sqrt{1+w^2}$; whereof the Fluent will be given, $\frac{\frac{1}{2}a^2}{v^2} = \frac{1}{2}w\sqrt{1+w^2} + \frac{1}{2}$ hyp. Log. $\overline{w+\sqrt{1+w^2}}$ ‡: Which corrected (by taking $v=c$ and $w=0$) becomes $\frac{\frac{1}{2}a^2}{v^2} - \frac{\frac{1}{2}a^2}{c^2} = \frac{1}{2}w\sqrt{1+w^2}$ $+ \frac{1}{2}$ hyp. Log. $\overline{w+\sqrt{1+w^2}}$. But, to ſhorten the re-

‡ Art. 126. and 281.

remaining Part of the Proceſs, let the latter Part of the Equation, or the Fluent of $\dot{w}\sqrt{1+w^2}$ be denoted by Q; then $\frac{aa}{2vv}$ being $=\frac{aa}{2cc}+Q$, we have $v=\frac{ac}{\sqrt{aa+2ccQ}}$; and conſequently $\frac{z}{d}$ $(=$hyp. Log. $\frac{c}{v})$ $=$ hyp. Log. $\frac{\sqrt{aa+2ccQ}}{a}=\frac{1}{2}$ hyp. Log. $\overline{1+\frac{2ccQ}{aa}}$. From which two Equations, the Velocity of the Ball, and the Diſtance it has moved, when its Direction makes any given Angle with the Horizon, may be computed, let the Medium be as denſe as it will: Alſo, from hence, if the Celerity anſwering to any one given Angle of Direction be known, the Celerity correſponding to any other given Direction may be found, together with the Diſtance deſcribed between the two Poſitions. For v (in the Deſcent of the Body) being, *univerſally*, equal to $\frac{ac}{\sqrt{aa+2c^2Q}}$, the Value of c, expreſſing the Celerity at the Vertex *A*, will be had from that Equation, and comes out $=\frac{av}{\sqrt{aa-2v^2Q}}$; whence alſo z $(=d\times$ hyp. Log. $\frac{c}{v})$ $=d\times$ hyp. Log. $\frac{a}{\sqrt{aa-2v^2Q}}=-\frac{1}{2}d\times$ hyp. Log. $\overline{1-\frac{2vvQ}{aa}}$. From which, the Celerity at A being known, the reſt is obvious.

But, in the aſcending Part of the Curve EA, both z and Q muſt be conſider'd as negative, or wrote with contrary Signs: And then, from the foregoing Equations, we ſhall alſo get $v=\frac{ac}{\sqrt{aa-2ccQ}}$, $c=\frac{av}{\sqrt{aa+2vvQ}}$, and

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and $-z = \frac{1}{2}d \times$ hyp. Log. $\overline{1 - \frac{2cc\mathcal{Q}}{aa}} = -\frac{1}{2}d \times$ hyp. Log. $\overline{1 + \frac{2vv\mathcal{Q}}{aa}}$; and, conſequently, $z = -\frac{1}{2}d \times$ hyp. Log. $\overline{1 - \frac{2cc\mathcal{Q}}{aa}} = \frac{1}{2}d \times$ hyp. Log. $\overline{1 + \frac{2vv\mathcal{Q}}{aa}}$ $= d \times$ hyp. Log. $\frac{v}{c}$: Anſwering in this Caſe.

It ſtill remains to take ſome notice of the Values of x and y (in order to have the Form, as well as the Length, of the Curve.) Theſe, indeed, are not ſo eaſy to bring out as That of z, given above; nor can they be exhibited in a general Manner, either by circular Arcs, or Logarithms (that I have been able to diſcover) but may, however, be approximated to any required Degree of Exactneſs, as will appear from what follows.

Since z $(= \text{AB})$ is found $= \frac{1}{2}d \times$ hyp. Log. $\frac{aa + 2c^2\mathcal{Q}}{aa}$, by taking the Fluxion thereof, we get $\dot{z} = \frac{ccd\dot{\mathcal{Q}}}{aa + 2cc\mathcal{Q}} = \frac{c^2d\dot{w}\sqrt{1+w^2}}{aa+2c^2\mathcal{Q}}$ (becauſe $\dot{\mathcal{Q}} = \dot{w}\sqrt{1+w^2}$) Therefore $\dot{y}$ $\left(= \frac{\dot{z}}{\sqrt{1+w^2}}\right) = \frac{c^2d\dot{w}}{aa + 2c^2\mathcal{Q}}$; and $\dot{x}$ $(= w\dot{y}) = \frac{c^2dw\dot{w}}{aa + 2c^2\mathcal{Q}}$: Which Equations, by taking r to 1, as a^2 to c^2 (or as the Square of the Force of Gravity to the Square of the Reſiſtance at A) are reduced to $\dot{y} = \frac{d\dot{w}}{r + 2\mathcal{Q}}$, and $\dot{x} = \frac{dw\dot{w}}{r + 2\mathcal{Q}}$: Whence we get $y = d$ into $\frac{w}{r + 2\mathcal{Q}} + \frac{\frac{2}{3} \times \overline{1 + ww}^{\frac{3}{2}} - \frac{2}{3}}{\overline{r + 2\mathcal{Q}}^2} +$

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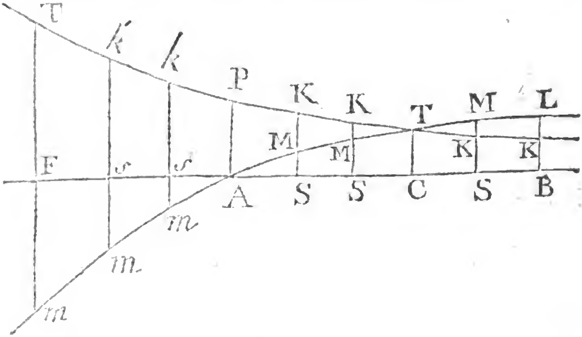
$$\frac{\frac{8}{3}w \times \overline{1+\frac{2}{3}w^2+\frac{1}{5}w^4} - \frac{8}{3}\mathcal{Q}}{\overline{r+2\mathcal{Q}}|^3}$$ &c. And $x = d$ into

$$\frac{\frac{1}{2}w^2}{r+2\mathcal{Q}} + \frac{\frac{1}{4}w \times \overline{1+ww}|^{\frac{3}{2}} - \frac{1}{4}\mathcal{Q}}{\overline{r+2\mathcal{Q}}|^2} +$$

$$\frac{\frac{1}{6} \times \overline{1+ww}|^3 - \frac{1}{6} - \frac{1}{2}\mathcal{Q}^2}{\overline{r+2\mathcal{Q}}|^3}$$ &c. Theſe Expreſſions (brought out by aſſuming $\frac{A}{r+2\mathcal{Q}} + \frac{B}{\overline{r+2\mathcal{Q}}|^2} +$ &c. for the Fluent ſought, and proceeding as in *Art.* 340.) converge very faſt when r is large in compariſon to $\mathcal{Q}$; but in other Caſes the required Values will be had, with leſs Trouble, from the following Method.



Let PKTK and AMTM be two Curves, whereof the Ordinates SK and SM, to the common *Abſciſſa* w (= AS) are expreſſed by $\frac{1}{r+2\mathcal{Q}}$ and $\frac{w}{r+2\mathcal{Q}}$ reſpectively: Then it is plain, from the foregoing Equations, that the Meaſures of the Areas of the ſaid Curves, multiply'd by d, will truly exhibit the Values of y and x; an-

anſwering to any given Value of w (or AS) the Tangent of the Angle of Direction; or, to ſpeak more geometrically, a Square upon AC (ſuppoſing AC = Radius = Unity) will be to either of the ſaid Areas ASKP, or ASM as the given Diſtance d, to the Value of y or x required——But now as to a Way for computing theſe Areas (without which what has been ſaid about them would be to very little Purpoſe) the Method of *Equi-diſtant Ordinates* may here be apply'd to very good Advantage (when the foregoing Serieſes do not converge) By means whereof the required Quantities may, with a little Trouble, be brought out to a ſufficient Degree of Exactneſs, let the Reſiſtance be as great as it will.

According to the ſame Way of proceeding, the Values of x and y, in the Aſcent of the Ball, will alſo be found, if the Ordinates sk and sm, generating the required Areas, be taken, every where, equal to $\frac{1}{r-2Q}$ and $\frac{w}{r-2Q}$ (inſtead of $\frac{1}{r+2Q}$ and $\frac{w}{r+2Q}$).

From what has been thus far delivered, it will not be very difficult to calculate (according to the foregoing Hypotheſis) all the principal Requiſites concerning the Motion and Track of a Ball in the Air, projected with a given Velocity, at a given Elevation; as will be more clearly ſeen by the Example ſubjoin'd.

Suppoſe a Cannon Ball of 4 Inches Diameter (whereof the Weight is nearly 9 Pounds) to be diſcharged at an Elevation of 45 Degrees, with a Velocity ſufficient to carry it to the Diſtance of one Mile, on the Plane of the Horizon, were it not for the Reſiſtance of the Air.——Then that Velocity, being the ſame as might be freely acquir'd in a perpendicular Deſcent of half a Mile *, will be found to anſwer to the Rate of 412 Feet, *per Second*, according to *Art.* 202. and 366. From whence it is alſo plain, that the Diſtance d (ſo often mention'd above) will here be expounded by 5333 Feet; and that the Celerity *(a)* with which the Reſiſtance would be equal to the Gravity (or Weight of the Ball) anſwers to the Rate of about 414 Feet *per Second.*

* Art. 366.

Moreover, ſince the Tangent of the Angle of Elevation, or the firſt Value of w, is given *equal to* Unity (or Radius) we have $\mathcal{Q}\ (\frac{1}{2}w\sqrt{w^2+1}+\frac{1}{2}\text{ hyp. Log. }\overline{w+\sqrt{w^2+1}}) = 1.1478$: From which, and $v\ (= 412\sqrt{\frac{1}{2}})$, we get $z\ (= \frac{1}{2}d \times \text{hyp. Log. } \overline{1+\frac{2vv\mathcal{Q}}{aa}}) = 2025$ Feet = the Arch deſcribed in the whole Aſcent. Alſo $c\ (= \frac{v}{\sqrt{1+\frac{2vv\mathcal{Q}}{aa}}}) = 199\frac{1}{3}$ Feet, for the Rate of the Velocity, *per Second*, at the higheſt Point: Whence $r\ (= \frac{aa}{cc}) = 4{,}314$; by Means whereof the greateſt Altitude of the Ball, and the horizontal Diſtance correſponding thereto will likewiſe be found: For let AF, *in the preceding Figure*, be taken $= 1$ (the given Value of w) and let the ſame be divided into three Parts by equi diſtant Ordinates (which Number will anſwer ſufficiently exact) then the ſucceſſive Values of w, for the Ordinates AP, *ks*, *ks* and TF, being $0, \frac{1}{3}, \frac{2}{3}$ and 1, *thoſe* of $\mathcal{Q}$ will be 0, 0.3394, 0.713, and 1.1478, and the Ordinates themſelves (or the correſponding Values of $\frac{1}{r-2\mathcal{Q}}$) = to 0.2318, 0.2751, 0.3463 and 0.4953, reſpectively. From whence, by adding the two Extremes to three times the Sum of the two middle Terms, and dividing the whole by 8, we get 0.3239 for the Value of a mean Ordinate †: Which, as AF is here equal to Unity, is alſo the Meaſure of the required Area AFTP: Which, therefore, being multiply'd by 5333 *(d)* gives 1727 Feet, for the horizontal Diſtance made good in the whole Aſcent. In the ſame Way the Area AF*m* is found $=0.1828$. Whence the greateſt Height of the Ball appears to be ($=0.1828\times5333$) $=975$ Feet.

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† *See* p. 117. *of my Mathematical Diſſertations.*

By taking $AC=1$, and repeating the Operation (only changing $r-2\mathcal{Q}$ to $r+2\mathcal{Q}$) the Area ACTP will come out $=0.1883$, and $ATC=0.0875$; which multiply'd by 5333 (as above) give 1004 F. and 467 F. for the Amplitude, and the Diſtance deſcended, from the higheſt Point, when the Direction of the Ball makes an Angle with the Horizon equal to that in which it was projected.

But, to have the Direction when the Ball ſtrikes the Ground, and the whole Amplitude of the Projection, we muſt find the Value of the Tangent AB, when the Area ABL is equal to (0.1828) the Area AF*m* (ſo that the Deſcent, from the higheſt Point, may become equal to the whole Aſcent.) In order thereto, let 0.0875 (ATC) be deducted from 0.1828 (AF*m*) and the Remainder 0.0953 will be $=$ CTBL; this, divided by TC (0.1513) quotes 0.63; which would be the Value of CB, if all the Ordinates CT, SM, *&c.* were equal: But, as it is obvious from the Nature of the Problem, and from the Law of the Ordinates already computed, that BL will be ſomething greater than CT, and conſequently CB leſs than 0.63——I therefore ſuppoſe the Value of CB may be about 0.56; and, accordingly, proceed to compute the Area of CBLT anſwering to this Number; by means of CT (0.1513) and BL (0.1852) and one intermediate Ordinate SM (0.1715) and find it (from the Approximation $\frac{CT+BL+4SM}{6}\times CB$) to come out $=0.0955$: Which is ſo near the required Value 0.0953, that it will be altogether needleſs to repeat the Operation. It is evident from hence, that the Tangent (AB) of the Angle of Direction, when the Ball ſtrikes the Ground, is 1.56; anſwering to 57°: 20′: From whence, CBKT being found $=0.0752$, the whole Area ABKF will be had $=0.2635$, and conſequently $0.2635\times5333=1405$ F. $=$ the Amplitude in the whole Deſcent.

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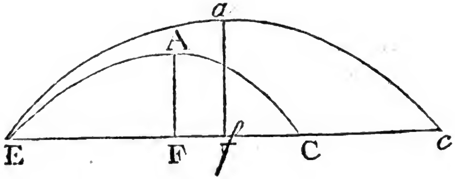
Furthermore, from the ſaid Value of w and that of c ($= 199\frac{1}{3}$) given above, we get z ($= \frac{1}{2} d \times$ hyp. Log. $\overline{1 + \frac{2ccQ}{aa}}$) $= 1788$ Feet, for the Arch deſcribed in the Deſcent; and alſo $v = 142\frac{1}{2}$ F. which multiply'd by 1.8527, the Secant of 57° : 20′, gives 264 F. for the Celerity of the Ball, *per Second*, at the End of its Flight.

Now, by collecting the principal of the foregoing Concluſions, it appears,



1°. That the Velocity at the higheſt Point A of the Trajectory will be at the Rate of $199\frac{1}{3}$ Feet, *per Second* : Which is to the Velocity at the higheſt Point *a* of the Parabola (E*ac*) that would be deſcribed, were it not for the Reſiſtance, as 2 to 3, nearly.

2°. EA=2025 and E*a*=3030
3°. EF=1727 and E*f*=2640
4°. AF= 975 and *af*=1320 } Feet.
5°. AC=1788 and *ac*=3030
6°. FC=1405 and *fc* =2640
7°. Angle C=57° : 20′ and *c* (=E) =45°.
8°. Velocity at C to that at E, as 264 to 412, or as 2 to 3, nearly.

Theſe Proportions, between the Diſtances, in Air and *in vacuo*, hold at an Elevation of 45°, when the Reſiſtance, at going off, is nearly equal to the Gravity, or Weight, of the Ball. If the Velocity be greater than that above ſpecified, or the Body, projected, be, either, leſs, or leſs denſe, the Curve will differ, *ſtill*, more from a Parabola.

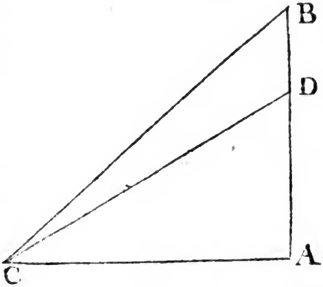
Hence it evidently appears, that the Effect of the Air's Reſiſtance upon very ſwift Motions, is too conſiderable to be intirely diſregarded in the Art of Gunnery.——'Tis true the Method given above is, by much, too intricate for common Practice; but when the Law of the Reſiſtance to very ſwift Motions is once ſufficiently eſtabliſhed (which, according to ſome late Experiments, ſeems to be in a Ratio greater than that of the Square of the Celerity) it will be no very difficult Matter to find out proper Approximations to correct the Proportions in common Uſe.

SECTION IX.

The Uſe of Fluxions in determining the Attraction of Bodies under different Forms.

PROB. I.

374. *Suppoſing* AC *perpendicular to* AB, *and that a Corpuſcle at* C *is attracted towards every Point or Particle of the Line* AB, *by Forces in the reciprocal duplicate Ratio of the Diſtances; to determine the Ratio of the whole Force whereby the Corpuſcle is urged in the Direction* CA.

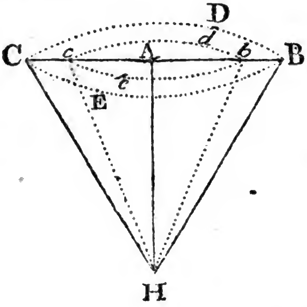


Put AC$=a$, and let AD (conſidered as variable by the Motion of D towards B) be denoted by x: Then, the Force of a Particle at D being as $\frac{1}{CD^2}$ (by Hypotheſis) its Efficacy in the propoſed Direction AC (will by the Reſolution of Forces) be as $\frac{AC}{CD} \times \frac{1}{CD^2} = \frac{AC}{CD^3} = \frac{a}{\overline{a^2+x^2}|^{\frac{3}{2}}}$: Therefore $\frac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$ is the Fluxion of the whole Force; whoſe Fluent, which (*by Art.* 85.) is $= \frac{x}{a \times \overline{a^2+x^2}|^{\frac{1}{2}}}$ $= \frac{AD}{CA \times CD}$, will, when AD=AB, be as the Force itſelf.

Q. E. I.

PROB. II.

375. *Suppoſing* BCDE *to repreſent a circular Plane, and that a Corpuſcle* H, *in the Axis thereof* AH, *is attracted by every Point or Particle of the Plane by Forces in the reciprocal duplicate Ratio of the Diſtances; to find the whole Force by which the Corpuſcle is urged towards the Plane.*



Let AH=a, and Hb=x; then Ab^2 $=x^2-a^2$; which multiply'd by (p= 3,14159 &c.) the Area of the Circle whoſe Radius is Unity, gives $p \times \overline{x^2-a^2}$ for the Area of the Circle A$cdbe$: whoſe Fluxion is $= 2px\dot{x}$. But the Force of a ſingle Particle at b, in the Direction HA, is as $\frac{AH}{Hb^3}$, or $\frac{a}{x^3}$ (ſee the laſt Problem) therefore the Fluxion of the whole Force is truly defined by $2px\dot{x} \times \frac{a}{x^3}$ or its Equal $\frac{2p\dot{x}}{x^2}$, and the Force itſelf by the Fluent of $\frac{2pa\dot{x}}{x^2}$; which (properly corrected) is $-\frac{2pa}{x} + \frac{2pa}{a} = 2p \times \overline{1 - \frac{a}{x}} = 2p \times \overline{1 - \frac{AH}{BH}}$, when $x = HB$. *Q. E. I.*

376. In the preceding Problems, we have ſuppoſed the Attraction of each Particle, to be as the Square of the Diſtance inverſely; that being the Law which is found to obtain in Nature: But if the Force, according to any other Law of Attraction, be required, the Proceſs will be very little different.

Thus, let the Attraction be as any Power *(n)* of the Diſtance: Then (in the laſt *Prob.)* the Force of a Particle at b (upon H) being as x^n, its Force in the Direction HA will be as $\frac{a}{x} \times x^n$ or ax^{n-1}; which multiply'd by $2px\dot{x}$ (as before) gives $2pax^n\dot{x}$: whereof the Fluent $\frac{2pax^{n+1} - 2pa^{n+2}}{n+1}$ $\left(= \frac{2p}{n+1} \times \overline{AH \times BH^{n+1} - AH^{n+2}}\right)$ will be as the Force required.

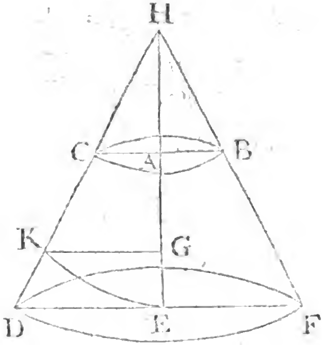
PROB. III.

377. *To determine the Attraction of a Cone* DHF *at its Vertex; the Attraction of each Particle being as the Square of the Diſtance inverſely.*

Put the Axis $EH = a$, the Length of the Slant-Side HD (or HF) $= b$, and AH (conſidered as variable) $= x$: Then *(by ſim. Triangles)* a (HE) : b (HF)

:: x

$:: x$ (HA) : HB $= \frac{bx}{a}$. But, by the laſt Problem,



the Attraction of all the Particles in the Circle BC will be meaſured by $2p \times \overline{1 - \frac{AH}{BH}} = 2p \times \overline{1 - \frac{a}{b}}$ (becauſe HB $= \frac{bx}{a}$): Which therefore being multiply'd by $\dot{x}$, and the Fluent taken, we thence have $x - \frac{ax}{b}$ for the Attraction of ACHB: And this, when $x = a$, will be, $2p \times \overline{EH - \frac{EH^2}{DH}}$, the Force of the whole Cone DEHF: Which, if HK be made = HE, and KG perpendicular to HE, will likewiſe be truly defined by $2p \times EG$ (becauſe HG $= \frac{EH^2}{DH}$). *Q. E. I.*

COROLLARY.

378. Seeing the Attraction of ACHB is, every where, as $x - \frac{ax}{b}$, or $\frac{b-a}{b} \times x$, it follows that the Forces of ſimilar Cones, at their Vertexes, are directly as their Altitudes.

PROB.

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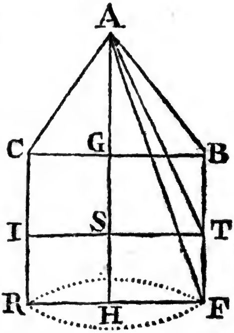
PROB. IV.

379. *To find the Force of a Cylinder* CBRF, *at any Point* A *in the produced Axis*; *the Law of Attraction being ſtill as in the preceding Problems.*

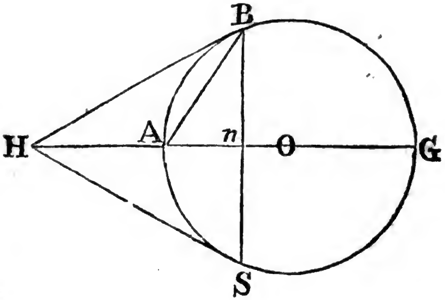


Put BG ($=$ CG $=$ RH) $= b$; and let AS (taken as variable) $= x$: Therefore AT$=\sqrt{b^2+x^2}$, and $2p \times \overline{1-\frac{AS}{AT}} = 2p \times \overline{1-\frac{x}{\sqrt{b^2+x^2}}}$: Which (*by Prob.* 2.) expreſſes the Force of all the Particles in the circular Surface IST.

Therefore $2p \times \overline{\dot{x}-\frac{x\dot{x}}{\sqrt{b^2+x^2}}}$ is the Fluxion of the required Force: Whoſe Fluent $(2p \times \overline{x-\sqrt{b^2+x^2}})$ when $x =$ AG, will be $= 2p \times \overline{AG-AB}$; but when $x =$ AH, it will be $= 2p \times \overline{AH-AF}$: Hence, by taking the former of theſe Values from the latter, we have $2p \times \overline{AB+BF-AF}$ for the Meaſure of the true Force by which a Corpuſcle at A is urged towards the Cylinder.

PROB. V.

380. *The Law of the Force being* ſtill *ſuppoſed the ſame; to determine the Attraction of a Sphere* OABGS, *at any given Point* H *above its Surface.*



Let BS be perpendicular to HG, and let HB be drawn; alſo put the Radius AO$=a$, OH$=b$, AH $(b-a)$ $=c$, H$n=y$, and HB$=c+x$; then A$n=y-c$, Gn $=2a-y+c$, and conſequently $\overline{y-c}\times\overline{2a-y+c}$ $(=$ $An\times Gn=Bn^2=BH^2-Hn^2)=\overline{c+x}|^2-y^2$: From which Equation we get $y=\frac{2ac+2c^2+2cx+x^2}{2a+2c}=$ $\frac{2bc+2cx+x^2}{2b}$ (becauſe $a+c=b$.) Whence alſo $2p\times$ $\overline{1-\frac{Hn}{HB}}$ * $=2p\times\overline{1-\frac{2bc+2cx+x^2}{2b\times\overline{c+x}}}=\frac{2p\times\overline{2ax-x^2}}{2b\times\overline{c+x}}$: Which multiply'd by $\frac{c\dot{x}+x\dot{x}}{b}=\dot{y}$ gives $\frac{p\times\overline{2ax\dot{x}-x^2\dot{x}}}{b^2}$ for the Fluxion of the required Force; whereof the Fluent

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* Art. 375.

$\frac{p \times \overline{ax^2 - \frac{1}{3}x^3}}{b^2}$ will be the Attraction of the Segment ABS: Which therefore, when B coincides with G and x is $= 2a$, becomes $\frac{4pa^3}{3b^2}$, for the Meaſure of the Attraction of the whole Sphere. *Q. E. I.*

COROLLARY I.

381. Hence the Attraction $\left(\frac{4pa^3}{3b^2}\right)$ at the Surface of the Sphere, where b is $= a$, will be $\frac{4pa}{3}$; and therefore is directly as the Radius of the Sphere.

COROLLARY II.

382. Since $\frac{4pa^3}{3}$ is known to expreſs the Content of Sphere whoſe Radius is a *, it is evident that the Attraction $\left(\frac{4pa^3}{3b^2}\right)$ of any Sphere is, univerſally, as its Quantity of Matter directly, and the Square of the Diſtance from its Center inverſely; and is, moreover, the very ſame as it would be, was all the Matter in the Sphere to be united in a Point at the Center.

* Art. 148.

COROLLARY III.

383. If inſtead of a Corpuſcle, or a ſingle Particle of Matter, at H, we ſuppoſe another Sphere, having its Center at H: Then, ſince the two Spheres, at O and H, act upon each other with the very ſame Forces, as if each Maſs was contracted into its Center, it follows that the abſolute Force, with which two ſpherical Bodies tend towards each other, is as the Product of their Maſſes directly, and the Square of the Diſtance of their

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Centers inverſely: And therefore, if the Maſſes are given, will be barely as the Square of the Diſtance.

PROB. VI.

384. *To determine the ſame as in the laſt Problem, the Force of each Particle being as any Power* (n) *of the Diſtance.*

Let HB $= x$, and let every thing elſe remain as above; then we ſhall have $y = \frac{c^2+2ac+x^2}{2b} = d + \frac{x^2}{2b}$ (by putting $d = \frac{c^2+2ac}{2b}$) and conſequently $\dot{y} = \frac{x\dot{x}}{b}$.

Now the Attraction of all the Particles in the circular Surface BS, is as $\frac{2p}{n+1} \times \overline{Hn \times HB^{n+1} - Hn^{n+2}}$ (*by Art.* 376.) $= \frac{2p}{n+1} \times \overline{yx^{n+1} - y^{n+2}}$: Which, multiply'd by $\dot{y}$, gives $\frac{2p}{n+1} \times \overline{x^{n+1} y\dot{y} - y^{n+2}\dot{y}}$ for the Fluxion of the required Force: Which, becauſe $y\dot{y}$ is $= \overline{d + \frac{x^2}{2b}} \times \frac{x\dot{x}}{b} = \frac{dx\dot{x}}{b} + \frac{x^3\dot{x}}{2b^2}$, will likewiſe be expreſſed by $\frac{2p}{n+1} \times \overline{\frac{dx^{n+2}\dot{x}}{b} + \frac{x^{n+4}\dot{x}}{2b^2} - y^{n+2}\dot{y}}$: Whereof the Fluent is $\frac{2p}{n+1} \times \overline{\frac{dx^{n+3}}{\overline{n+3} \times b} + \frac{x^{n+5}}{\overline{n+5} \times 2b^2} - \frac{y^{n+3}}{n+3}}$: Which, when B coincides with A, or $x = y = c$, will be $= \frac{2p}{n+1} \times \overline{\frac{dc^{n+3}}{\overline{n+3} \times b} + \frac{c^{n+5}}{\overline{n+5} \times 2b^2} - \frac{c^{n+3}}{n+3}}$: But, when B co-

B coincides with G, or $x = y = 2a + c$ $(= f)$ it will become $= \frac{2p}{n+1} \times \overline{\frac{df^{n+3}}{\overline{n+3} \times b} + \frac{f^{n+5}}{\overline{n+5} \times 2b^2} - \frac{f^{n+3}}{n+3}}$: Therefore the Difference of theſe two, which is $=$

$$\frac{2pf^{n+3}}{n+1} \times \frac{\overline{n+5} \times \overline{2bd - 2b^2} + \overline{n+3} \times f^2}{\overline{n+3} \times \overline{n+5} \times 2b^2} - \frac{2pc^{n+3}}{n+1} \times$$

$$\frac{\overline{n+5} \times \overline{2bd} - \overline{2b^2} - \overline{n+3} \times c^2}{\overline{n+3} \times \overline{n+5} \times 2b^2} =$$

$$\frac{\overline{\overline{1+n} \times ab - cc} \times 2pf^{n+3} + \overline{\overline{5+n} \times ab + cc} \times 2pc^{n+3}}{\overline{n+1} \times \overline{n+3} \times \overline{n+5} \times b^2}$$

(becauſe $f = a + b$, and $2db = c^2 + 2ac$) will be the Attraction of the whole Sphere. *Q. E. I.*

COROLLARY.

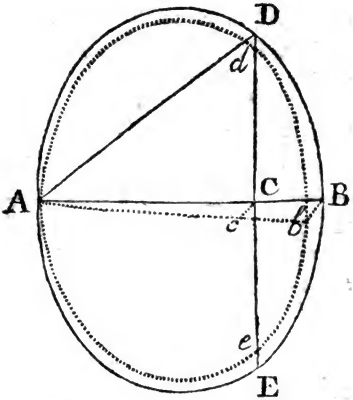
385. Hence, the Attraction at the Surface of the Sphere (where $c = 0$) will be $\frac{2p}{n+1} \times \frac{\overline{1+n} \times \overline{2a}^{n+3} + \overline{n+5} \times 0^{n+3}}{\overline{n+1} \cdot \overline{n+3} \cdot \overline{n+5}}$: Which, if $n + 3$ be poſitive, will be $= \frac{2p \times \overline{2a}^{n+3}}{\overline{n+3} \times \overline{n+5}}$; but, otherwiſe, infinite.

PROB. VII.

386. *Suppoſing* ADB*b*A *to be a* Cuneus *of uniformly denſe Matter, compriz'd by two equal and ſimilar eliptic Planes* ADBEA *and* A*dbe*A, *inclin'd to each other, at the common Vertex* A, *of either their firſt or ſecond Axes, in an indefinitely ſmall Angle* BA*b*; *To determine the Attraction thereof at the Point* A, *ſuppoſing the Force of each Particle of Matter to be as the Square of the Diſtance inverſely.*

Let

Let DE be any Ordinate to the Axis AB, and let AD be drawn; alſo put AB$=a$, BC$=x$, CD$=y$, and the Sine of the Angle B Ab, formed by the two Planes (to the Radius 1) $=d$; and let the Equation of either Curve be $y^2=fx-x^2-gx^2$: Which will anſwer, to the Conjugate, or Tranſverſe Axis thereof, according as the Value of g is poſitive or negative.



Now it will be, 1 (Radius) : d :: $a-x$ (AC) : Cc $=d\times\overline{a-x}$, the Thickneſs of the *Cuneus* at the Ordinate (or Section) DE: Moreover, becauſe $AD^2=AC^2+CD^2$, we have $AD=\sqrt{\overline{a-x}^2+fx-x^2-gx^2}$: Whence, $\frac{DE\times Cc}{AC\times AD}$, expreſſing (*by Art.* 374.) the Attraction of the Particles in the indefinitely narrow Rectangle DE$\times$Cc, will be defined by $\frac{2d\sqrt{fx-x^2-gx^2}}{\sqrt{\overline{a-x}^2+fx-x^2-gx^2}}$: Which therefore, multiply'd by $\dot{x}$, will give the Fluxion of the Force to be found. But when $fx-x^2-gx^2$ be-

becomes $= 0$, x will be $= \frac{f}{1+g}$ ($=$AB) $= a$; therefore, by ſubſtituting for f, our Fluxion will be tranſformed to

$$\frac{2d\dot{x}\sqrt{\overline{1+g}\times ax - \overline{1+g}\times x^2}}{\sqrt{\overline{a-x}^{2} + \overline{1+g}\times ax - \overline{1+g}\times x^2}} = \frac{2d\dot{x}\sqrt{\overline{1+g}\times \overline{ax-x^2}}}{\sqrt{\overline{a-x}^{2}+\overline{1+g}\times \overline{ax-x^2}}} = \frac{2d\dot{x}\sqrt{\overline{1+g}\times x}}{\sqrt{a-x+\overline{1+g}\times x}} =$$

$$\frac{2d\times\overline{1+g}^{\frac{1}{2}}\times x^{\frac{1}{2}}\dot{x}}{\overline{a+gx}^{\frac{1}{2}}} = \frac{\overline{1+g}^{\frac{1}{2}}\times 2dx^{\frac{1}{2}}\dot{x}}{a^{\frac{1}{2}}} \times \overline{1-\frac{gx}{2a}+\frac{3g^2x^2}{2.4a^2}-\frac{3\cdot 5g^3x^3}{2.4.6a^3}\ \&c.}$$

Whereof the Fluent, when $x = a$, will be $\overline{1+g}^{\frac{1}{2}}\times 2ad\times \overline{\frac{2}{3}-\frac{2}{5}\times\frac{g}{2}+\frac{2}{7}\times\frac{3g^2}{2.4}-\frac{2}{9}\times\frac{3\cdot 5g^3}{2.4.6}\ \&c.}$ Which, becauſe $\overline{1+g}^{\frac{1}{2}}\times a$ is $= f\times\overline{1+g}^{-\frac{1}{2}} = f\times \overline{1-\frac{g}{2}+\frac{3g}{2.4}-\frac{3\cdot 5g^2}{2.4.6}\ \&c.}$ will (by multiplying the two Serieſes together &c.) be reduced to $2df\times \overline{\frac{2}{3}-\frac{2.4g}{3\cdot 5}+\frac{2.4\cdot 6g^2}{3\cdot 5\cdot 7}-\frac{2\cdot 4.6.8g^3}{3\cdot 5\cdot 7\cdot 9}\ \&c.}$

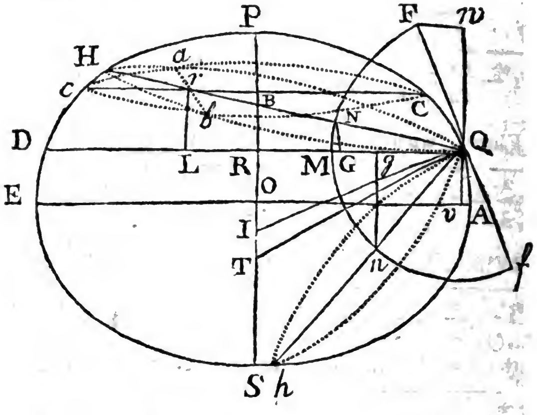
Q. E. I.

It may be obſerved, that the Fluent given above may be brought out without an Infinite Series (*by Art*, 126. *and* 278.) But the Solution here exhibited is beſt adapted to what follows hereafter; to which the Propoſition itſelf is premiſed as a *Lemma*.

PROB. VIII.

387. *To determine the Attraction at any Point* Q *in the Surface of a given Spheroid* OAPES.

Let QRL be perpendicular to the Axis PS of the Spheroid, and QT perpendicular to the Tangent F*f* of the generating Ellipſis at Q, meeting PS in T: Moreover, let Q*a*H*b* be a Section of the Spheroid by a Plane perpendicular to that of the Ellipſis APES, and thro' any Point *r*, in the Axis thereof, draw CB*c* and *r*L parallel to AE and PS: And make the Abſciſſa Q*r* $= x$, its correſponding Semi-Ordinate *ra* (or *rb*) $= y$, QR $= a$,



and RT $= b$; alſo let the Sine (NG) of the Angle HQD (to the Radius NQ $= 1$) $= p$, its Co-ſine QG $= q$, and the Ratio of OA² to OP², as any given Quantity *h* to Unity. Now, by reaſon of the ſimilar Triangles Q*r*L and QNG, we have *r*L (BR) $= px$, and QL $= qx$, and therefore B*r* (RL) $= qx - a$: Alſo, from the Nature of the Ellipſis, $AO^2 : PO^2$ $(h : 1)$ :: RT (b) : OR $= \frac{b}{h}$: Likewiſe $AO^2 : PO^2$

$(h : 1)$

$(b:1)::\mathrm{QR}^2:\mathrm{OP}^2-\mathrm{OR}^2$; and $\mathrm{PO}^2:\mathrm{AO}^2\ (1:b)$ $::\mathrm{OP}^2-\mathrm{OB}^2:\mathrm{BC}^2=b\times\overline{\mathrm{OP}^2}\,\overline{-\mathrm{OB}^2}=b\times$ $\overline{\mathrm{OP}^2-\overline{\mathrm{OR}+\mathrm{RB}}}^{\,2}=b\times\overline{\mathrm{O}r^2-\mathrm{OR}^2-2\mathrm{OR}\times\mathrm{RB}-\mathrm{RB}^2}$ $=\mathrm{QR}^2+b\times\overline{-2\mathrm{OR}\times\mathrm{RB}-\mathrm{RB}^2}$; becauſe (by the former Proportion) $\mathrm{QR}^2=b\times\overline{\mathrm{OP}^2-\mathrm{OR}^2}$: Whence, by the Property of the Circle, C*acb*, we get $y^2\ (\mathrm{BC}^2-\mathrm{B}r^2)=\mathrm{QR}^2-$ $\mathrm{B}r^2-b\times\overline{2\mathrm{OR}\times\mathrm{RB}+\mathrm{RB}^2}=a^2-\overline{qx-a}^{\,2}-b\times$ $\overline{\frac{2b}{b}\times px+p^2x^2}=\overline{aq}\,\overline{-bp}\times 2x-\overline{q^2+bp^2}\times x^2$: Which Equation, by making $1+B=b$, becomes $y^2=$ $\overline{aq}\,\overline{-bp}\times 2x-\overline{q^2+p^2+Bp^2}\times x^2=\overline{aq-bp}\times 2x-x^2-$ Bp^2x^2 (becauſe $q^2+p^2=1=\mathrm{QN}^2$: Which being only of two Dimenſions, the Curve Q*a*H*b*, whereto it belongs, is an Ellipſis.

The Equation of the Curve Q*a*H*b* being now obtained, let its Axis QH be ſuppoſed to revolve about Q, as a Center (the Plane of the Curve being always perpendicular to that of the Ellipſis APES) and let the Fluxion of the Arch MN (expreſſing the Angle deſcribed from the time the ſaid Axis begins its Motion at the Poſition ALD) be denoted by $\dot{A}$: Then, it is evident from the preceding Problem, that, $\overline{2aq-2bp}\times 2\dot{A}\times$ $\overline{\frac{2}{3}-\frac{2\cdot 4Bp^2}{3\cdot 5}+\frac{2\cdot 4\cdot 6B^2p^4}{3\cdot 5\cdot 7}}$ &c. will be the Fluxion of the Attraction of the correſponding Part DQH of the Solid, upon a Corpuſcle at Q, conſidered as acting in the Direction HQ (which Expreſſion is found, by, barely, writing $2aq-2bp$, $\dot{A}$, and Bp^2, in the ſaid Problem, for f, d, and g reſpectively.)

Hence, by the Reſolution of Forces, the Fluxion of the Attraction, in the Directions QR and Q*w* (perpendicular to QR) will be truly exhibited by

$$\overline{2aq-2bp}\times 2\dot{A}q\times\overline{\frac{2}{3}-\frac{2\cdot 4Bp^2}{3\cdot 5}+\frac{2\cdot 4\,6B^2p^4}{3\cdot 5\cdot 7}}\ \&c.\ \text{and}$$

$$\overline{2aq-2bp}\times 2\dot{A}p\times\overline{\frac{2}{3}-\frac{2\cdot 4Bp^2}{3\cdot 5}+\frac{2\cdot 4\cdot 6B^2p^4}{3\cdot 5\cdot 7}}\ \&c.$$

Let now another Plane Q*h* be ſuppoſed to revolve about the Point Q, the contrary Way to the former, from QD towards Q*f*; and let *(ng)* the Sine of the Angle RQ*h* be denoted by P, and its Co-ſine (Q*g*) by $\mathcal{Q}$: Then the Fluxion of the Attraction of the Part DQ*h*, in the foreſaid Directions QR and Q*w* (by writing $-P$ inſtead of p and $\mathcal{Q}$ inſtead of q) will appear to be

$$\overline{2a\mathcal{Q}+2bP}\times 2\dot{A}\mathcal{Q}\times\overline{\frac{2}{3}-\frac{2\cdot 4BP^2}{3\cdot 5}+\frac{2.4.6B^2P^4\ \&c.}{3\cdot 5\cdot 7}}$$

and

$$\overline{2a\mathcal{Q}+2bP}\times -2\dot{A}P\times\overline{\frac{2}{3}-\frac{2\cdot 4BP^2}{3\cdot 5}+\frac{2\cdot 4\cdot 6B^2P^4}{3\cdot 5\cdot 7}}\ \&c.$$

Which being added to thoſe of the former Part, in the ſame Directions, and $\frac{\dot{p}}{q}$ and $\frac{\dot{P}}{\mathcal{Q}}$ reſpectively ſubſtituted inſtead of $\dot{A}$ *, we have

$$4a\text{ into }\frac{2}{3}\times\overline{q\dot{p}+\mathcal{Q}\dot{P}}-\frac{2\cdot 4B}{3\cdot 5}\times\overline{qp^2\dot{p}+\mathcal{Q}P^2\dot{P}}\ \&c.$$

$$+\,4b\ \text{ into }\ \frac{2}{3}\times\overline{P\dot{P}-p\dot{p}}-\frac{2\cdot 4B}{3\cdot 5}\times\overline{P^3\dot{P}-p^3\dot{p}}\ \&c.$$

And

$$4a\text{ into }\frac{2}{3}\times\overline{p\dot{p}-P\dot{P}}-\frac{2\cdot 4B}{3\cdot 5}\times\overline{p^3\dot{p}-P^3\dot{P}}\ \&c.$$

$$-4b\text{ into }\frac{2}{3}\times\overline{\frac{p^2\dot{p}}{q}+\frac{P^2\dot{P}}{\mathcal{Q}}}-\frac{2\cdot 4B}{3\cdot 5}\times\overline{\frac{p^4\dot{p}}{q}+\frac{P^4\dot{P}}{\mathcal{Q}}}\ \&c.$$

for

* Art 142.

for the Fluxion of the Attraction of both Parts together in the foreſaid Directions: Whereof the Fluents, when N coincides with F, and n with f, will be the Attraction of the whole Spheroid in thoſe Directions. But now, in order to determine theſe Fluents with as little Trouble as poſſible, let m be aſſumed to denote any whole poſitive Number; then the Fluent of $\frac{p^{2m}\dot{p}}{q}$, or $\frac{p^{2m}\dot{p}}{\sqrt{1-p^2}}$, will be univerſally $=\frac{-q}{2m}\times\overline{p^{2m-1}}$

$$\overline{+\frac{2m-1}{2m-2}\times p^{2m-3}+\frac{2m-1\,.\,2m-3}{2m-2\,.\,2m-4}\times p^{2m-5}}\ (m)$$

$$+\frac{1\cdot 3\cdot 5\ldots\overline{2m-1}}{2\cdot 4\cdot 6\ldots 2m}\times \text{the Arch } (MN) \text{ whoſe Sine}$$

is p[*]: And that of $\frac{P^{2m}\dot{P}}{Q}$, or $\frac{P^{2m}\dot{P}}{\sqrt{1-P^2}}$ (in the ſame Manner) $=\frac{-Q}{2m}\times\overline{P^{2m-1}+\frac{2m-1}{2m-2}\times P^{2m-3}}$, &c. $+\frac{1\cdot 2\cdot 3\ldots\overline{2m-1}}{2\cdot 4\cdot 6\ldots 2m}\times$ Arch (Mn) whoſe Sine is P. But when N coincides with F, and n with f, the Sines p and P, of the Arches MF and Mf, becoming equal, and (the Co-ſine) $Q=-$ (Co-ſine) q, it is evident that the Sum of the Fluents of $\frac{p^{2m}\dot{p}}{q}$ and $\frac{P^{2m}\dot{P}}{Q}$, will, in that Caſe, be truly exhibited by $\frac{1\cdot 3\cdot 5\ldots\overline{2m-1}}{2\cdot 4\cdot 6\ldots 2m}\times MF+\frac{1\cdot 3\cdot 5\ldots\overline{2m-1}}{2\cdot 4\cdot 6\ldots 2m}\times$ Mf, or its Equal $\frac{1\cdot 3\cdot 5\ldots\overline{2m-1}}{2\cdot 4\cdot 6\ldots 2m}\times$ FMf; becauſe

[*] Art. 296.

cauſe, then all the reſt of the Terms (by reaſon of the equal Quantities P, p and Q, $-q$) deſtroy one another. After the ſame Manner the Sum of the Fluents of $qp^{2m}\dot{p}$ and $QP^{2m}\dot{P}$, in the foreſaid Circumſtance, will

* Art. 297. appear to be $= \dfrac{1\cdot3\cdot5\cdot7\ldots\overline{2m-1}}{2\cdot4\cdot6\cdot8\ldots\overline{2m+2}}\times FMf$ *.

Now, to apply this to the Matter in hand, let the Exponent of B, in any Term of either of the above found Fluxions be, univerſally, expreſſed by n; then the numeral Coefficient (annexed to B) will be defined by $\dfrac{2\cdot4\cdot6\ldots\overline{2n+2}}{1\cdot3\cdot5\ldots\overline{2n+3}}$, and the variable Quantities multiplied thereby, in the firſt Line of the former Fluxion, will be $qp^{2n}\dot{p}+QP^{2n}\dot{P}$: Therefore $\dfrac{2\cdot4\cdot6\ldots\overline{2n+2}}{1\cdot3\cdot5\ldots\overline{2n+3}}\times B^{n}\times\overline{qp^{2n}\dot{p}+QP^{2n}\dot{P}}$ is a General Term, (from whence, if n be expounded by 1, 2, 3 &c. ſucceſſively, that whole Line will be produced.) But, the Fluent of $qp^{2n}\dot{p}+QP^{2n}\dot{P}$, in the Circumſtance above ſpecified, (putting $m=n$ and $FMf=k$, appears to be $=\dfrac{1\cdot3\cdot5\cdot7\ldots\overline{2n-1}}{2\cdot4\cdot6\cdot8\ldots\overline{2n+2}}\times k$: Which, therefore, multiply'd by $\dfrac{2\cdot4\cdot6\ldots\overline{2n+2}}{3\cdot5\ldots\overline{2n+3}}$ $\times B^{n}$, gives $\dfrac{1\cdot3\cdot5\cdot7\ldots\overline{2n-1}}{2\cdot4\cdot6\cdot8\ldots\overline{2n+2}}\times\dfrac{2\cdot4\cdot6\ldots\overline{2n+2}}{3\cdot5\ldots\overline{2n+3}}$ $\times B^{n}k=\dfrac{B^{n}k}{\overline{2n+1}\times\overline{2n+3}}$, for the true Fluent of the ſaid General Term: Which, if n be expounded by

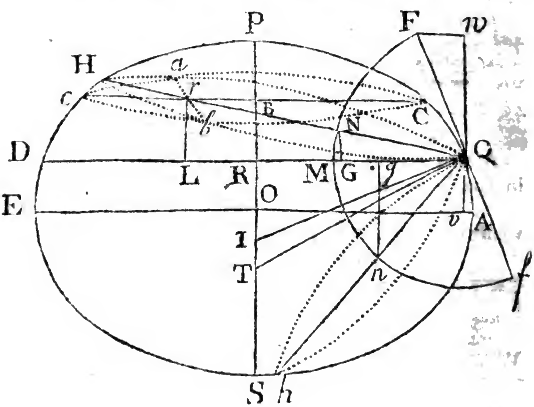
0,

0, 1, 2, 3 &c. ſucceſſively, will become equal to $\frac{k}{1 \cdot 3}$, $\frac{Bk}{3 \cdot 5}$, $\frac{B^2 k}{5 \cdot 7}$, $\frac{B^3 k}{7 \cdot 9}$ &c. reſpectively; and therefore the Fluent of the whole Line (drawn into the general Multiplicator $4a$) is $= 4ak \times \overline{\frac{1}{1 \cdot 3} - \frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7} - \frac{B^3}{7 \cdot 9}}$ &c. But now, for the Fluent of the ſecond Line: This, it is plain, will be $= 4b$ into $\frac{2}{3} \times \overline{\frac{P^2}{2} - \frac{p^2}{2}} - \frac{2 \cdot 4B}{3 \cdot 5} \times \overline{\frac{P^4}{4} - \frac{p^4}{4}}$ &c. Which, in the foreſaid Circumſtance, when $P = p$, intirely vaniſhes. Therefore it appears, that the Attraction of the whole Spheroid, in the Direction QR, is truly expreſſed by $4ak \times \overline{\frac{1}{1 \cdot 3} - \frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7} - \frac{B^3}{7 \cdot 9}}$, or its Equal $4k \times \overline{\frac{1}{1 \cdot 3} - \frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7}}$ &c. $\times \mathcal{QR}$.

After the ſame Manner the Fluent of the firſt Line, in the latter of our two Fluxions, will be found to vaniſh: And $\frac{2 \cdot 4 \cdot 6 \ldots \overline{2n+2}}{1 \cdot 3 \cdot 5 \ldots \overline{2n+3}} \times B^n \times \overline{\frac{p^{2n+2}\dot{p}}{q} + \frac{P^{2n+2}\dot{P}}{\mathcal{Q}}}$ will be a General Term to the ſecond Line. Whereof the Fluent (by expounding $2m$ by $2n+2$) appears, from above, to be $= \frac{2 \cdot 4 \cdot 6 \ldots\ldots \overline{2n+2}}{3 \cdot 5 \cdot 7 \ldots\ldots \overline{2n+3}} \times B^n k \times \frac{1 \cdot 3 \cdot 5 \ldots \overline{2n+1}}{2 \cdot 4 \cdot 6 \ldots \overline{2n+2}} = \frac{B^n k}{2n+3}$: Which, when

n is interpreted by 0, 1, 2, 3 &c. ſucceſſively, comes out equal to $\frac{k}{3}$, $\frac{Bk}{5}$, $\frac{B^2k}{7}$ &c. reſpectively: Therefore the Attraction of the Spheroid, in the Direction Qw, is exhibited by $-4bk \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}}$ &c. and conſequently, That in the oppoſite Direction Qv, by $4bk \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}} = 4k \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}\ \&c.} \times RT = 4k \times \overline{1+B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}\ \&c.} \times OR$ (becauſe $\overline{1+B} \times OR = RT$.)

From which and the Force in the Direction QR (found above) not only the Direction of the abſolute



Attraction, but that Attraction itſelf will be known: For, let RI be taken to QR, as the Force in the Direction Qv to that in the Direction QR; and then, by the Compoſition of Forces, QI will be the Direction of the Attraction, or the Line in which a Corpuſcle at Q tends to deſcend: And the Attraction itſelf, in that Direction, (being to that in QR, as QI to QR) will be defined by $4k \times \overline{\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}}\ \&c. \times QI$; which, ſince $4k$ is conſtant, will alſo be as $\overline{\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}}\ \&c. \times QI$. *Q. E. I.*

Corollary.

388. Since, by Conſtruction, RI : QR :: $\overline{1+B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}}\ \&c. \times OR : \overline{\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}}\ \&c. \times QR$, it follows that $\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}\ \&c. : \overline{1+B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}}\ \&c. :: RO : RI$; whence (by Diviſion) $\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}\ \&c. : \frac{3B}{3\cdot5} - \frac{3B^2}{5\cdot7} + \frac{3B^3}{7\cdot9}\ \&c. :: OR \left(: \frac{OT}{B} \right) : OI$; and conſequently, $\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}\ \&c. : 3 \times \overline{\frac{1}{3\cdot5} - \frac{B}{5\cdot7} + \frac{B^2}{7\cdot9}}\ \&c. :: OT : OI$.

Hence it appears that the Direction QI, of the abſolute Attraction, divides the Part of the Axis OT, intercepted by the Center and Normal, in a given Ratio: And that the Attraction itſelf (being de-

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fined by $\overline{\frac{1}{1\cdot 3} - \frac{B}{3\cdot 5} + \frac{B^2}{5\cdot 7} \text{ \&c.} \times QI}$) is every where as the ſaid Line of Direction QI.

SCHOLIUM.

389. Although the foregoing Concluſions are exhibited by Infinite Serieſes, yet the Sums of thoſe Serieſes are explicable by means of the Arch of a Circle.

Thus, let the Series $\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}$ &c. (which is one of the two original ones above found) be put $=S$, and let $B = t^2$; then by Subſtitution, and multiplying the whole Equation by t^3, we ſhall have $\frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7}$ &c. $= t^3S$, and conſequently $t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7}$ &c. $= t - t^3S$: Where, the former Part of the Equation is known to expreſs the Arch of a Circle, whoſe Tangent is t $(B^{\frac{1}{2}})$ and Radius Unity *: Wherefore, putting that Arch $= A$, we have $A = t - t^3S$, and conſequently $S = \frac{t - A}{t^3} = \frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}$ &c.

Moreover, ſince it appears that

$$\left.\begin{array}{l} \frac{B}{3} - \frac{B^2}{5} + \frac{B^3}{7} \text{ \&c.} \\[2ex] - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9} \text{ \&c.} \end{array}\right\} \text{is} = \frac{2B}{3\cdot 5} - \frac{2B^2}{5\cdot 7} + \frac{2B^3}{7\cdot 9} \text{ \&c.}$$

(where the Sum of $\frac{B}{3} - \frac{B^2}{5} + \frac{B^3}{7}$ &c. is already found $= \frac{t-A}{t^3} \times B = \frac{t-A}{t}$, and where That

of

* Art. 142.

of $-\frac{B}{5}+\frac{B^2}{7}$ &c. by the ſame Method wlll come out $=\frac{t-A-\frac{1}{3}t^3}{t^3}$) it is evident that $\frac{2B}{3\cdot 5}-\frac{2B^2}{5\cdot 7}+\frac{2B^3}{7\cdot 9}$ &c. $=\frac{t-A}{t}+\frac{t-A-\frac{1}{3}t^3}{t^3}=\frac{\frac{2}{3}t^3+t-A\times\overline{1+t^2}}{t^3}$; and conſequently $\frac{1}{1\cdot 3}-\frac{B}{3\cdot 5}+\frac{B^2}{5\cdot 7}$ &c. $\left(=\frac{1}{3}-\frac{\frac{2}{3}t^3+t^2-A\times\overline{1+t^2}}{2t^3}\right)=\frac{A\times\overline{1+t^2}-t}{2t^3}$: Which is the Value of the other original Series found above: From whence that of $\frac{3}{3\cdot 5}-\frac{3B}{5\cdot 7}+\frac{3B^2}{7\cdot 9}$ will alſo be had $=\frac{3t+2t^3-3A\times\overline{1+t^2}}{2t}$.

Hence, if

$$\frac{t-A}{t^3}\left(=\frac{1}{3}-\frac{B}{5}+\frac{B^2}{7}-\frac{B^3}{9}\right)\text{ be put }=f;$$

$$\frac{A\times\overline{1+t^2}-t}{2t^3}\left(=\frac{1}{1\cdot 3}-\frac{B}{3\cdot 5}+\frac{B^2}{5\cdot 7}\text{ \&c.}\right)=g;$$

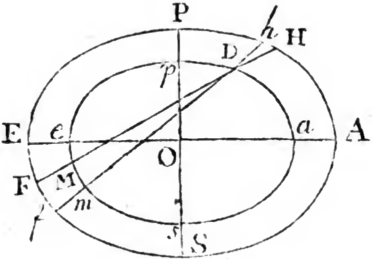
And

$$\frac{3t+2t^3-3A\times\overline{1+t^2}}{2t}\left(=\frac{3}{3\cdot 5}-\frac{3B}{5\cdot 7}+\frac{3B^2}{7\cdot 9}\text{ \&c.}\right)=h$$

it is evident that OT will be to OI, in the conſtant Ratio of g to h; and that the Forces in the Directions QI, QR, and Qv, will be as $g\times$QI, $g\times$QR, and $f\times\overline{1+B}\times$OR reſpectively: Where $1+B$ is $=\frac{AO^2}{PO^2}$.

PROB. IX.

390. *To determine the Attraction at any Point* D *within a given Spheroid* OAPES.



Let O*apes* be another Spheroid, concentric with, and ſimilar to, the given one; whoſe Surface D*e*M &c. paſſes thro' the given Point D; alſo let FD*f* and HD*h* be taken as two oppoſite, indefinitely ſlender, Cones (or Pyramids) conceived to be form'd by drawing innumerable Lines HDF, *h*D*f* &c. through the common Vertex D) which Cones (or Pyramids) having the ſame Angle, may be conſidered as ſimilar; and ſo their Forces, at D, will be as the Altitudes DF and DH*: And, therefore, the Exceſs of the former, above the latter, or the Force whereby a Corpuſcle at D, tends towards F, through the, contrary, Action of the two oppoſite Cones, will be as DF—DH, or as DM; becauſe *(by the Property of the Ellipſis)* MF is, in all Poſitions, equal to DH.

Hence it appears that the Parts of Matter FM*mf* and HD*h*, without the Spheroid *apes* (acting equally, in contrary Directions) can have no Effect at D; And this, being every where the Caſe, the whole, efficacious, Force at D muſt therefore be that of the Spheroid O*apes*.

Hence, if the Ratio of Oa^2 to Op^2 (or of OA^2 to OP^2) be denoted by that of $1+B$ to 1, as in the laſt Problem, it

* Art. 378.

it follows, from thence, that the Attraction at D, in the Directions DM and DN (perpendicular to PS and AE; *ſee the next Fig.*) will be expounded by $\overline{\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}}$ &c. × DM, and $\overline{1+B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}}$ &c. × DN reſpectively, or by their Equals $g \times \text{DM}$ and $f \times \overline{1+B} \times \text{DN}$: Where the Values of f and g are the ſame as given in the preceding Article.

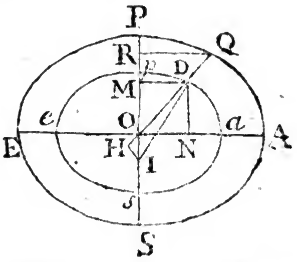
COROLLARY.

391. Hence the Force wherewith a Corpuſcle, any where within a given Spheroid, is attracted, either, towards the Axis, or the Plane of its Equator, is directly as the Diſtance therefrom.

PROB. X.

392. *Suppoſing every Particle of Matter in a Spheroid to have a Tendency to recede, both, from the Axis* PS, *and from the Plane of the greateſt Circle, by Means of Forces that are as the Diſtances from the ſaid Axis, and Plane, reſpectively; to find the Direction* DI *wherein a Corpuſcle, at any Point* D, *tends to move through the Action of the ſaid Forces and the Attraction conjunctly; and likewiſe the whole compound Force in that Direction.*

Let DM and DN be perpendicular to PS and AE, and let the given Forces, in the Direction of thoſe Lines (independent of the Attraction) be expreſſed by $m \times \text{DM}$ and $n \times \text{DN}$ reſpectively.



Therefore, ſince (by the laſt Problem) the Force of Attraction in the ſaid Directions is defined by $g\times$DM and $f\times\overline{1+B}\times$ DN, the whole reſulting Forces will be truly denoted by $\overline{g-m}\times$DM, and $\overline{f\times\overline{1+B}-n}\times$ DN: Whence (by the Compoſition of Forces) it will be, $g-m : f\times\overline{1+B}-n$ :: DN (OM) : MI; whence the Point I is given:

Alſo DM : DI :: $\overline{g-m}\times$ DM (the Force in the Direction DM) : $\overline{g-m}\times$DI, the Force in DI. *Q.E.I.*

PROB. XI.

393. *Every thing being ſuppoſed as in the preceding Problems, 'tis required to determine the Force of all the Particles in the Line (or Column)* QDO *tending to the Center* O *of the Spheroid.*

Let IH be perpendicular to QO produced (*ſee the laſt Fig.*) then the abſolute Force, in the Direction DI, being $\overline{g-m}\times$ DI, that in the Direction DH, whereby a Corpuſcle at D is urged towards the Center, will be $\overline{g-m}\times$ DH. Let now OD (conſidered as variable) be denoted by x; then becauſe the Ratio of OM to MI is given (being every where as $g-m$ to $f\times\overline{1+B}-n$, *by the Precedent*) and the Triangles ODM and IOH are ſimilar, it follows that the Ratio of OD to OH will be given, or conſtant; and conſequently that of DH to OH, likewiſe: Let therefore this Ratio of DH to OH be expreſſed by that of r to s, and we ſhall have DH$=\frac{rx}{s}$, and conſequently ($\overline{g-m}\times$ DH) the Force at D, equal to $\overline{g-m}\times\frac{rx}{s}$: Which therefore being multi-

ply'd

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ply'd by $\dot{x}$, and the Fluent taken, there comes out $\frac{\overline{g-m}\times rx^2}{2s}=\frac{g-m}{2}\times \text{DO}\times \text{DH}$, for the whole Force of the Line or Column OD at the Center.

Q. E. I.

COROLLARY.

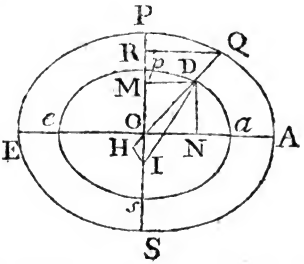
394. If the given Forces m and n be ſuch that the Ratio of OM to MI, (which is found to be univerſally as $g-m$ to $f\times\overline{1+B}-n$) may become as $1:1+B$ (or as $p\text{O}^2:a\text{O}^2$) it is evident (from the Property of the Ellipſis) that the Line of Direction DI will be always perpendicular to the Surface of the Spheroid *Oapes*. In which Caſe OD × DH is alſo (by the Nature of the Ellipſis) $=\text{O}a^2$: And therefore the Force $\left(\frac{g-m}{2}\times\text{OD}\times\text{DH}\right)$ of OD is $=\frac{g-m}{2}\times\text{O}a^2$: Which, when D coincides with Q, will become $\frac{g-m}{2}\times\text{AO}^2$; and is, therefore, a conſtant Quantity.

Moreover, ſince in this Caſe, $g-m:f\times\overline{1+B}-n::1:1+B$ (by Hypotheſis) we have $m-\frac{n}{1+B}=g-f$: Which Equation, if n be taken $=0$, gives $m=g-f=\frac{2B}{3\cdot5}-\frac{4B^2}{3\cdot7}+\frac{6B^3}{7\cdot9}$ &c. $=\frac{\overline{3+t^2}\times A-3t}{2t^3}$*; * Art. 389.

But, if m be taken $=0$, it will then give $n=-\overline{1+B}\times\overline{g-f}=-\overline{1+B}\times\overline{\frac{2B}{3\cdot5}-\frac{4B^2}{5\cdot7}+\frac{6B^3}{7\cdot9}}$ &c. Where, $t=B^{\frac{1}{2}}$, and $A=$ the Arch whoſe Tangent is t, and Radius Unity.

PROB. XII.

395. *If an oblate Spheroid* OAPES, *whereof the Square of the Equatoreal Diameter* AE, *is to that of the Axis* PS, *in any given Ratio of* $1 + B$ *to* 1, *revolves about its Axis, in ſuch a Time, that the centrifugal Force, at the Equator* A, *is to the Attraction at the Surface of a Sphere whoſe Radius is* OA, *in the Ratio of* $\frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} + \frac{6B^3}{7 \cdot 9}$ &c. to $\frac{1}{3}$: *I ſay, in that Caſe, every Particle of the Spheroid will be* in Equilibrio; *ſo that, tho' the Coheſion of the Parts was to ceaſe, the Figure itſelf would remain unchanged.*



For, the Attraction of the Spheroid, at A, being defined by $\overline{\frac{1}{1 \cdot 3} - \frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7}}$ &c. $\times$ AO (*Art.* 387.) it is evident (by conceiving $B = 0$) that $\frac{AO}{3}$ will repreſent the Attraction at the Surface of the Sphere whoſe Radius is AO: Whence (by Hypotheſis) the centrifugal Force at A (putting $m = \frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} + \frac{6B^3}{7 \cdot 9}$ &c.) will be truly defined by $m \times$ AO; and conſequently That, at any other Point D, by $m \times DM$ (becauſe the centrifugal Forces of Bodies deſcribing unequal Circles, in equal Times, are known to be directly as the Radii *.) Hence, and from the Corollary to the laſt Problem, it appears that the Direction of Gravitation DI is always perpendicular to the Surface *apes*; and that the Force of all the Particles in the Line (or Canal) OD or OQ, towards the Center O, will continue invariable, take the Point Q in what Part of the Arch APE you will: From which laſt Conſideration, it follows that the Force, or Preſſure of every Canal QO, at the Center O, (conſidering the Body in a fluid State) will be the ſame: Whence (by the Principles of Hydroſtatics) a Corpuſcle at D has no Tendency to move, either Way, in the Line OQ: And therefore, as it hath no Tendency to move in the Direction of the Surface D*pe* (the Gravitation being perpendicular thereto) it is evident, *from Mechanics*, that no Motion at all can enſue, in any Direction. *Q. E D.*

* Art. 213.

Corollary I.

396. Since m is $= \frac{2B}{3\cdot 5} - \frac{4B^2}{5\cdot 7} + \frac{6B^3}{7\cdot 9}$ &c. the Gravitation $(\overline{g - m} \times DI)$ at any Point D in the Spheroid will therefore be as $\overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} \&c.} \times DI = \frac{t-A}{t^3} \times DI$ (*ſee Art.* 389.)

Corollary II.

397. If the Time of Revolution be given $= p$, and q be put to denote the Time wherein a (ſolid) Sphere, of the ſame Denſity with the Spheroid, muſt revolve; ſo that the centrifugal Force, at the Equator thereof, may be equal to the Gravity: Then, as this laſt Time is known to continue the ſame, whatever the Magnitude of that Sphere is ‡; and the centrifugal Forces, in equal

‡ Art. 213. and 381.

Circles, are alſo known to be inverſely as the Squares of the periodic Times——it follows, that $p^2 : q^2 :: \frac{1}{3}AO$ (the Attraction, or centrifugal Force, reſpecting the Sphere OA, revolving in the Time q) : $\overline{\frac{2B}{3\cdot 5} - \frac{4B^2}{5\cdot 7} + \frac{6B^3}{7\cdot 9}}$ &c. × AO, the centrifugal Force of the Spheroid at A; revolving in the Time p. From which Proportion we get $\frac{q^2}{3p^2} = \frac{2B}{3\cdot 5} - \frac{4B^2}{5\cdot 7} + \frac{6B^3}{7\cdot 9}$ &c. $= \frac{\overline{3+t^2} \times A - 3t}{2t^3}$ (*Art.* 394.) Whence, by Help of the Trigonometrical-Canon, the Value of t $(= B^{\frac{1}{2}})$ and, conſequently, the Ratio of the two principal Diameters, will be found; ſo that all the Parts of the Spheroid may remain *in Equilibrio*. But, when $\frac{q^2}{3p^2}$ is ſmall, the Solution by an Infinite Series is preferable: For, then the Series $\frac{2B}{3\cdot 5} - \frac{4B^2}{5\cdot 7}$ &c. $(= \frac{q^2}{3p^2})$ converging ſufficiently ſwift, we ſhall, by the Reverſion thereof, find $B = \frac{5q^2}{2p^2} + \frac{25 \times 6q^4}{4 \times 7p^4} + \frac{125 \times 37q^6}{8 \times 49p^6}$ &c. In which Caſe the Ratio of the Equatoreal Diameter to the Axis, if we take only the firſt Term of the Series, will be, as $\sqrt{1 + \frac{5q^2}{2p^2}} : 1$, or as $1 + \frac{5q^2}{4p^2}$; nearly.

Which, if $\frac{p^2}{q^2} = 289$, or the centrifugal Force at the Equator be to the Gravity as 1 to 289 (that being the Proportion at the Equator of the Earth *) will come out as 231 to 230.

* Art. 217.

Co-

COROLLARY III.

398. Becauſe, $\frac{\overline{3+t^2}\times A-3t}{2t^3}$, the latter Part of our foregoing Equation will be equal to Nothing, both when t is Nothing and Infinite, it is evident that the Value thereof cannot, in any intermediate Circumſtance of t, exceed a certain aſſignable Quantity.

Wherefore, to determine this Limit of the Value of $\frac{q^2}{3p^2}$ (beyond which the Problem becomes impoſſible) let the Fluxion of $\frac{\overline{3+t^2}\times A-3t}{2t^3}$, or its Double $\frac{\overline{3+t^2}\times A}{t^3}-\frac{3}{t^2}$ be taken and put $=0$, and you will have $-\overline{9+t^2}\times A\dot{t}+\overline{3t+t^3}\times\dot{A}+6t\dot{t}=0$: Which, becauſe $\dot{A}=\frac{\dot{t}}{1+t^2}$ * will be reduced to $9t+7t^3-\overline{1+t^2}\times\overline{9+t^2}\times A=0$; where t is found $=2,5293$, from whence the correſponding Values of $\sqrt{\overline{1+t^2}}$, and $\frac{q}{p}$ come out $=2,7198$, and 0.5805 &c. reſpectively. Hence it appears that it is impoſſible for the Parts of the Spheroid, in a fluid State, to continue at Reſt among themſelves, when the Time of Revolution is ſo great that $\frac{q}{p}$ exceeds $0,5805$ &c. And that, of all the Spheroids which can be aſſumed by a Fluid revolving about an Axis, That whoſe Equatoreal Diameter is to its Axis as 2,7198 to Unity, will perform its Revolutions in the ſhorteſt Time.

Thus, for Example, if a (ſolid) Sphere of the ſame common Denſity with the Earth was to revolve about its Axis in the Time of 84¾ Minutes, the centrifugal Force

* Art. 142.

Force at the Equator thereof would, it is known, be
* Art. 217. equal to the Gravity*: Therefore, by taking $\frac{84\frac{3}{4}}{p}\left(=\frac{q}{p}\right)$ $= 0{,}5805$ *&c.* the Time p will come out $= \overset{M}{146}$ or $\overset{H}{2}\ \overset{M}{26}$. Which Time is the leaſt, poſſible, wherein a Fluid, of the ſame common Denſity with the Earth, can revolve, ſo as to preſerve its ſpheroidal Figure. And this holds univerſally, let the Magnitude of the Body, or Fluid, be what it will.

COROLLARY IV.

399. Hence alſo may be determined the Spheroid, which a ſpherical Body (of Ice or any other Matter) revolving in a given Time s, will converge to, when reduced to a fluid State.

For, ſince the Momenta of Rotation, in equal Spheres and Spheroids, are to one another, in a Ratio compounded of the direct Ratio of their Equatoreal Diameters, and the inverſe Ratio of the Times of their Rotation, it follows, if d be put $=$ the Diameter of the given Sphere, and $E =$ the Equatoreal Diameter of the required Spheroid, that $\frac{d}{s} = \frac{E}{p}$ (becauſe the Quantity of Motion about the Axis is not affected by the Action of the Particles one upon another, while the Figure of the Fluid is changing.) Moreover, ſince the Maſſes of the Sphere and Spheroid are alſo equal to each other (by Hypotheſis) we have d^3 $(=AE^2 \times PS) = \frac{E^3}{\overline{1+t^2}|^{\frac{1}{2}}}$: From which two Equations, exterminating d, there ariſes $p = \overline{1+t^2}|^{\frac{1}{6}} \times s$, for the Time of Revolution of the required Spheroid: Whence, by ſubſtituting this Value of p in the general Equation $\frac{q^2}{3p^2}$ $=$

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$= \frac{\overline{3+t^2}\times A-3t}{2t^3}$, we get $\frac{q^2}{3s^2} = \overline{1+t^2}|^{\frac{1}{3}} \times \frac{\overline{3+t^2}\times A-3t}{2t^3}$; from the Solution of which the Value of t, and the Spheroid itſelf, will be given.

But, ſince the Value of the latter Part of the Equation can never exceed a certain aſſignable Quantity, the Matter propoſed can therefore be only poſſible under certain Limitations: In order to determine theſe Limitations, let the Fluxion of $\overline{1+t^2}|^{\frac{1}{3}} \times \frac{\overline{3+t^2}\times A-3t}{2t^3}$ be taken and put $=0$, and it will be found that $\overline{t^4+24t^2+27}\times A-15t^3-27t=0$: Whence t comes out $=7.5$, and the correſponding Value of $\frac{q}{s}=0{,}927$, nearly.

Hence the Parts of the Fluid cannot poſſibly come to an Equilibrium among themſelves, when the Time s is leſs than $\frac{q}{0{,}927}$, but will continue to recede from the Axis, *in Infinitum.*

If q be taken $= \overset{M}{84\tfrac{3}{4}}$ (as in the Example to the preceding Corollary) s will be equal $\overset{M}{91} = \overset{H}{1} : \overset{M}{31}$. From which it appears, that, if the Earth (or a ſpherical Body of the ſame Denſity) was to revolve about its Axis in leſs than $\overset{H}{1} : \overset{M}{31}$; and, in the mean time, be reduced to a State of Fluidity, the Parts thereof towards the Equator would aſcend, and continue to recede from the Axis, *in Infinitum.*

COROLLARY V.

400. Seeing the Values of t and A are given when the Spheroid is given, it follows that the Gravi-

tation $\left(\frac{t-A}{t^3}\times QI\right)$ at any Point in the Surface of a Spheroid, whereof the Parts are kept *in Equilibrio*, by their Rotation about the Axis, will be accurately as a Perpendicular to the Surface at that Point, continued to the Axis of the Figure. Therefore the Gravitation at the Equator is to that at either of the Poles, as the Equatoreal Diameter to the Axis inverſly.

COROLLARY VI.

401. But, if the Spheroid differs but little from a Sphere, the Exceſs of QI above AO will (by the Property of the Ellipſis) be nearly as OR^2. Whence it appears that the Increaſe of Gravitation, in going from the Equator to the Pole, is as the Square of the Sine of Latitude, nearly.

COROLLARY VII.

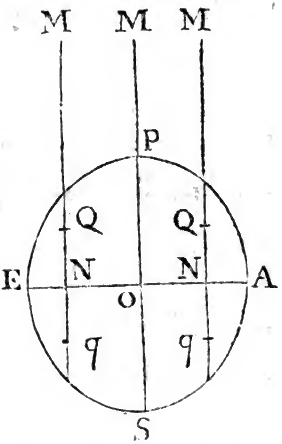
402. Moreover, ſince the Ratio of the Equatoreal Diameter to the Axis is found, in this Caſe, to be that of $1+\frac{5q^2}{4p^2}$ to 1 †, the Exceſs of that Diameter above the Axis will be to the Axis as $\frac{5q^2}{4p^2}$ to Unity; that is, as $\frac{5}{4}$ of the centrifugal Force at the Equator to the mean Force of Gravity. Whence, as the centrifugal Forces, in unequal Circles, are univerſally as the Radii directly, and the Squares of the periodic Times inverſly, it follows that the foreſaid Exceſs (in Figures nearly ſpherical) will be as the Radii directly, and as the Denſity and the Square of the Time of Rotation inverſly: From which Proportions, the Ratios of the greateſt and leaſt Diameters of the Planets may be inferred from each other; ſuppoſing the Times of their Rotation, about their Axes, to be known.

† Art. 397.

PRO-

PROB. XIII.

403. *To determine the Figure which a Fluid will acquire when, beſides the mutual Gravitation of the Parts thereof, it is attracted by another Body, ſo remote, that all Lines drawn from it to the Surface of the Fluid, may be taken as Parallels.*



Let OAPES be the propoſed Fluid, and let MPS and MQq be Right-lines, drawn from the remote Body *M*; whereof the former MPS paſſes thro' the Center of Gravity O: Moreover, let the Plane AE be perpendicular to the Axis MOS; and put NQ$=a$ and OM (the Diſtance of the remote Body) $=d$; alſo put the Semi-diameter of the Body (at M) $=r$, and let its Denſity be to that of the Fluid APES, as any Quantity v to Unity. Then ſince, according to the foregoing Calculations, the Attraction at the Surface of a Sphere (of a given Denſity) is expreſſed by $\frac{1}{3}$ of the Radius, it follows that the Attraction of the Body *M*, at its Surface, will be explicable by $\frac{vr}{3}$: And therefore, the Force varying according to the Square of the Diſtance inverſly *, it will be, d^2 (MN^2) $: r^2 :: \frac{vr}{3} : \frac{vr^3}{3d^2}$, the Attraction of *M*, at the Diſtance MN: Alſo $\overline{d-a}^2$ (MQ^2) $: r^2 :: \frac{vr}{3} : \frac{vr^3}{3\times\overline{d-a}^2}$, its Attraction at the

Diſtance

* Art. 382.

Diſtance MQ. Whence the Difference of theſe two, or $\frac{vr^3}{3\times\overline{d-a}|^2}-\frac{vr^3}{3d^2}$ ($=\frac{vr^3}{3d^3}\times\overline{2a+\frac{3a^2}{d}+\frac{4a^3}{d^2}}$ *&c.*) will be as the Force whereby a Corpuſcle at Q endeavours to recede from the Plane AE: Which becauſe (by Hypotheſis) d is very great in reſpect of a, will (by rejecting all the Terms after the firſt) be expreſſed by $\frac{2vr^3}{3d^3}\times a$, or its Equal $\frac{2vr^3}{3d^3}\times NQ$.

In the very ſame Manner, the Force whereby a Corpuſcle at q, below the Plane AE, tends to recede therefrom, will be defined by $\frac{2vr^3}{3d^3}\times Nq$.

Now, therefore, ſeeing theſe Forces are, every where, as the Diſtances NQ, Nq, from the Plane AE, it appears (*by Art.* 393. *and* 394.) that the Figure OAPES will be a Spheroid; whereof the Equation, for the Relation of its two principal Diameters (putting $n=\frac{2vr^3}{3d^3}$) is $n=-\overline{1+B}\times\overline{\frac{2B}{3\cdot5}-\frac{4B^2}{5\cdot7}+\frac{6B^3}{7\cdot9}}$ *&c.* (In which, the Ratio of PS^2 to AE^2 is denoted by that of 1 to $1+B$.) Hence, by reverting the Series, we have $B=-\frac{15n}{2}-\frac{225n^2}{28}$ *&c.* and conſequently PS : AE :: 1 : $\sqrt{1-\frac{15n}{2}-\frac{225n^2}{28}}$ *&c.* :: 1 : $1-\frac{15n}{4}$, nearly: Which, by reſtoring the Value of n, becomes PS : AE :: 1 : $1-\frac{5vr^3}{2d^3}$.

Q. E. I.

Co-

COROLLARY.

404. Becauſe $\frac{r}{d}$ expreſſes the Sine of the apparent Semi-diameter of the Body *M*, to the Radius 1) ſeen at the Diſtance OM, it follows, if the ſaid Sine be denoted by c, that PS : AE :: $1 : 1 - \frac{5v}{2} \times c^3$; and conſequently, by Diviſion, PS : PS — AE :: $1 : \frac{5v}{2} \times c^3$.

Hence it appears, that the Forces of the Planets, to produce Tides at the Earth's Surface, are to one another as their Denſities, and the Cubes of their apparent Diameters conjunctly. (For the Sines of ſmall Arcs are nearly as the Arcs themſelves.)

EXAMPLE.

405. If c be taken = the Sine of 16′ (expreſſing the mean Apparent Semi-diameter of the Moon) and $v = \frac{5}{4}$ (the Ratio of her Denſity with reſpect to that of the Earth) our laſt Proportion will become PS : PS — AE :: 1 : 0,000000315 : Whence, if PS be taken = 42000000 Feet (the Meaſure of the Earth's Diameter) PS — AE will come out $= \overset{F}{13{,}23}$.

SECT.

SECTION X.

Of the Application of FLUXIONS *to the Reſolution of ſuch Kinds of Problems* DE MAXIMIS ET MINIMIS, *as depend upon a particular Curve, whoſe Nature is to be determined.*

I SHALL begin this Section with premiſing the following uſeful

THEOREM.

406. *If the Relation of two flowing Quantities y and u be required; ſo that, when the Fluent of* $y^m\dot{u}$ *becomes equal to a given Value, that of* $\frac{y^r \times \overline{\dot{u}\dot{u} \pm \dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$ *may be a* Maximum *or a* Minimum; *I ſay, their Relation muſt be ſuch that* $\frac{y^{r-m}\dot{u} \times \overline{\dot{u}\dot{u} \pm \dot{y}\dot{y}}|^{n-1}}{\dot{y}^{2n-1}}$ *may be, every where, the ſame, or equal to a conſtant Quantity.*

The Demonſtration hereof depends upon the ſubſequent

LEMMA.

407. If $a\alpha + b\beta = Q$, wherein α and β are indeterminate, the Value of $A \times \overline{\alpha\alpha \pm pp}|^n + B \times \overline{\beta\beta + pp}|^n$ will be a *Maximum* or *Minimum*, when $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$ and $\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$ are equal to each other

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other. For, by taking the Fluxions of both Expreſſions we have $a\dot{\alpha} + b\dot{\beta} = 0$, and $2nA\alpha\dot{\alpha} \times \overline{\alpha\alpha \pm pp}|^{n-1} + 2nB\beta\dot{\beta} \times \overline{\beta\beta \pm pp}|^{n-1} = 0$: From whence, $\dot{\alpha}$ and $\dot{\beta}$ being exterminated, there reſults $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1} = \frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$. *Q. E. D.*

Hence, if $a\alpha + b\beta + c\gamma + d\delta$ *&c.* $= Q$ (where α, β, γ, δ *&c.* are indeterminate) it follows that $A \times$ $A \times \overline{\alpha\alpha \pm pp}|^{n} + B \times \overline{\beta\beta \pm pp}|^{n} + C \times \overline{\gamma\gamma \pm pp}|^{n} + D \times \overline{\delta\delta \pm pp}|^{n}$ *&c.* will be a *Maximum* or *Minimum*, when all the Quantities $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$, $\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$, $\frac{C\gamma}{c} \times \overline{\gamma\gamma \pm pp}|^{n-1}$ *&c.* are equal to each other. For that Expreſſion is a *Maximum* (or *Minimum)* when it cannot be increaſed (or decreaſed) by altering the Values of the indeterminate Quantities involved therein; but it may be increaſed (or decreaſed) by altering only two of them (as α and β) whilſt the reſt remain unchanged; unleſs $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$ and $\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$ are equal to each other. (This is proved above.) Therefore, when $A \times \overline{\alpha\alpha \pm pp}|^{n} + B \times \overline{\beta\beta \pm pp}|^{n} + C \times \overline{\gamma\gamma \pm pp}|^{n} +$ *&c.* is a *Maximum* or *Minimum*, the Quantities $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$ and $\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$ cannot be unequal: And, by the very ſame Argument, no other two of the Quantities above ſpecified can be unequal.

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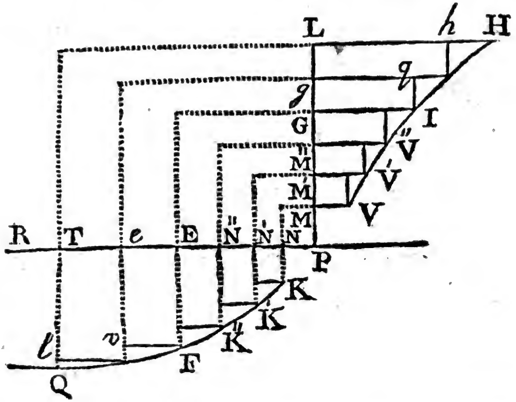
If, in the Right-line PR, there be now aſſumed $N\overset{\prime}{N} = \alpha$, $\overset{\prime}{N}\overset{\prime\prime}{N} = \beta$, *&c.* and upon theſe, as Baſes,



Rectangles $\overset{\prime}{N}K$, $\overset{\prime\prime}{N}\overset{\prime}{K}$ be ſuppoſed, whoſe Altitudes NK, $\overset{\prime}{N}\overset{\prime}{K}$ *&c.* are denoted by a, b, c, d *&c.* it is evident that $a\alpha + b\beta + c\gamma + d\delta$ *&c.* ($= Q$) will be expreſſed by the Sum of all the ſaid Rectangles, or the whole Polygon N*l*.

Moreover, if, in the Right-line PL (perpendicular to PR) there be taken $M\overset{\prime}{M}$, $\overset{\prime}{M}\overset{\prime\prime}{M}$ *&c.* each equal to p, and, upon theſe equal Baſes, Rectangles $M\overset{\prime}{V}$, $\overset{\prime\prime}{M}\overset{\prime}{V}$ *&c.* be conſtituted, whoſe Altitudes are denoted by $A \times \frac{\overline{\alpha\alpha \pm pp}\big|^{n}}{p^{2n}}$, $B \times \frac{\overline{\beta\beta \pm pp}\big|^{n}}{p^{2n}}$, *&c.* it is likewiſe plain that the Value of $\frac{A \times \overline{\alpha\alpha \pm pp}\big|^{n}}{p^{2n-1}} + \frac{B \times \overline{\beta\beta \pm pp}\big|^{n}}{p^{2n-1}} + \frac{C \times \overline{\gamma\gamma \pm pp}\big|^{n}}{p^{2n-1}}$ will be truly repreſented by the whole Polygon Mh. Which Polygon (as p is conſtant) will be a *Maximum* or *Minimum*, when $A \times \overline{\alpha\alpha \pm pp}^{n} + B \times \overline{\beta\beta \pm pp}^{n} +$ &c. is a *Maximum* or *Minimum*; that is, when all the Quantities $\frac{A\alpha}{a} \times \frac{\overline{\alpha\alpha \pm pp}^{n-1}}{p^{2n-1}}$, $\frac{B\beta}{b} \times \frac{\overline{\beta\beta \pm pp}^{n-1}}{p^{2n-1}}$, &c. are equal to each other (as has been proved above.)

Let now, A, B, C, D &c. be expounded by any Powers, (MP^{r}, $\overset{\prime}{\mathrm{M}}\mathrm{P}^{r}$, $\overset{\prime\prime}{\mathrm{M}}\mathrm{P}^{r}$ &c.) of the reſpective Diſtances from a given Point P; and let, at the ſame time, the correſponding Values of a, b, c, d &c. be interpreted by any other propoſed Powers MP^{m}, $\overset{\prime}{\mathrm{M}}\mathrm{P}^{m}$, $\overset{\prime\prime}{\mathrm{M}}\mathrm{P}^{m}$ &c. of the ſame given Diſtances: Then the Area of the Polygon Nl will be expreſſed by $\mathrm{MP}^{m} \times \alpha + \overset{\prime}{\mathrm{M}}\mathrm{P}^{m} \times \beta + \overset{\prime\prime}{\mathrm{M}}\mathrm{P}^{m} \times \gamma$ &c. $(= \mathcal{Q})$; and that of the Polygon Mh, by $\mathrm{MP}^{r} \times \frac{\overline{\alpha\alpha \pm pp}^{n}}{p^{2n-1}} + \overset{\prime}{\mathrm{M}}\mathrm{P}^{r} \times \frac{\overline{\beta\beta \pm pp}^{n}}{p^{2n-1}} + \overset{\prime\prime}{\mathrm{M}}\mathrm{P}^{r} \times \frac{\overline{\gamma\gamma \pm pp}^{n}}{p^{2n-1}} +$ &c. And the foreſaid equal Quantities $\frac{A\alpha}{a} \times \frac{\overline{\alpha\alpha \pm pp}^{n-1}}{p^{2n-1}}$, $\frac{B\beta}{b} \times \frac{\overline{\beta\beta \pm pp}^{n-1}}{p^{2n-1}}$ &c. will become $\mathrm{MP}^{r-m} \times \frac{\alpha \times \overline{\alpha\alpha \pm pp}^{n-1}}{p^{2n-1}}$, $\overset{\prime}{\mathrm{M}}\mathrm{P}^{r-m} \times \frac{\beta \times \overline{\beta\beta \pm pp}^{n-1}}{p^{2n-1}}$, &c. reſpectively.

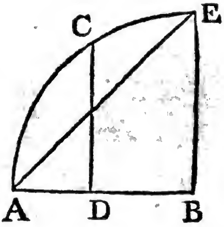
Now let the Number of the Rectangles be ſuppoſed indefinitely great, and their Breadths indefinitely ſmall, ſo that the Area of each of the two Polygons N*l* and M*h* may be taken for that of its circumſcribing Curve: Moreover, let *u* and *y* be put to repreſent the Diſtances of any two correſponding Ordinates EF and GI from the given Point P; and let $\dot{x}$ be every where expreſſed by p ($=M'M=M'M''=$ &c.) Then, $\dot{u}$ being a general Value for any of the Quantities α, β, γ, δ &c. (or NN', $N'N''$ &c.) it follows; Firſt, that the Fluxion of the Area of the Curve NEFK (the Ordinate being, every where, $=y^m$) will be truly defined by $y^m\dot{u}$; Secondly, that the Fluxion of the Area MGIV (by ſubſtituting y, u and $\dot{y}$ inſtead of their Equals) will be $\dfrac{y^r\times\overline{uu\pm\dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$; and, laſtly, that the Value of each of the equal Quantities, $MP^{r-m}\times\dfrac{\alpha\times\overline{\alpha\alpha\pm pp}|^{n-1}}{p^{2n-1}}$, $M'P^{r-1}\times\dfrac{\beta\times\overline{\beta\beta\pm pp}|^{n-1}}{p^{2n-1}}$, &c. above ſpecified, will be expreſſed by $\dfrac{y^{r-m}\times\dot{u}\times\overline{\dot{u}\dot{u}\pm\dot{y}\dot{y}}|^{n-1}}{\dot{y}^{2n-1}}$. *Whence the Theorem is manifeſt.*

408. If *R* and *S* be aſſumed to denote any Functions of *y* (that is, any two Quantities expreſſed in Terms of *y* and given Coefficients;) then, in order to have the Fluent of $S\times\dfrac{\overline{\dot{u}\dot{u}\pm\dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$ a *Maximum* or *Minimum*, when that of $R\dot{u}$ becomes equal to a given Value, it is requiſite that $\dfrac{S\dot{u}}{R}\times\dfrac{\overline{\dot{u}\dot{u}\pm\dot{y}\dot{y}}|^{n-1}}{\dot{y}^{2n-1}}$ ſhould be a conſtant

Quantity: This, alſo, is evident from the preceding Demonſtration; and may be of Uſe when the above premiſed Theorem is not ſufficiently general.

PROB. I.

409. *To determine the Nature of the Curve* ACE; *ſo that, the Length of the Arch* AE *being given, the Area* ABE *ſhall be a* Maximum.



Calling (as uſual) the Abſciſſa AD, x; the Ordinate DC, y; and the Arch AC, z, we have $\dot{x} = \sqrt{\dot{z}^2 - \dot{y}^2}$ *; and therefore $y\dot{x}$ † $= y \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}^{\,\frac{1}{2}} =$ the Fluxion of the Area ADC. Now, ſince, by the Queſtion, the Fluent of $y \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}^{\,\frac{1}{2}}$ is to be a *Maximum*, when That of $\dot{z}$ becomes equal to a given Quantity (ACE) let theſe two Fluxions be, reſpectively, compared with $\frac{y^r \times \overline{\dot{u}\dot{u} - \dot{y}\dot{y}}^{\,n}}{\dot{y}^{2n-1}}$ and $y^m \dot{u}$ (as given in the foregoing Theorem*.) By which means, $n = \frac{1}{2}$, $r = 1$, $\dot{u} = \dot{z}$, and $m = 0$; and conſequently $\frac{y^{r-m} \dot{u} \times \overline{\dot{u}\dot{u} - \dot{y}\dot{y}}^{\,n-1}}{\dot{y}^{2n-1}} = y\dot{z} \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}^{\,-\frac{1}{2}}$: Which (according to the ſaid Theorem) being, every where, equal to a conſtant Quantity, we ſhall, by putting that Quantity $= a$, and ordering the Equation, get $\dot{z}^2 = \frac{a^2\dot{y}^2}{a^2 - y^2}$, and $\dot{x}$ $(\sqrt{\dot{z}^2 - \dot{y}^2}) = \frac{y\dot{y}}{\sqrt{a^2 - y^2}}$; and, conſequently, (by taking

* Art. 135.

† Art. 112.

* Art 406.

taking the Fluent) $x = a - \sqrt{a^2 - y^2}$, or $2ax - xx = y^2$; which is the common Equation of a Circle.

Q. E. I.

COROLLARY.

410. It follows from hence, that the greateſt Area that can poſſibly be contain'd by a Right-line (AE) joining two given Points, and any Curve-line ACE of a given Length, terminating in the ſame Points, will be when the ſaid Curve-line is an Arch of a Circle.

PROB. II.

411. *The Length of the Arch* AE *(ſee the preceding Figure) being given, to determine the Nature of the Curve, ſo that the Solid generated by the Rotation thereof may be a* Maximum.

* Art. 145. Since the Fluent of $y^2 \times \overline{\dot{z}^2 - \dot{y}^2}|^{\frac{1}{2}}$ ($= y^2\dot{x}$ *) is required to be a *Maximnm*, when that of $\dot{z}$ has a given Value ACE, every thing will remain as in the laſt Problem; *only*, r muſt here be $= 2$: And therefore (*by the Theorem*) we have $y^2\dot{z} \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}|^{-\frac{1}{2}} = a$. Whence $\dot{z} = \dfrac{a\dot{y}}{\sqrt{a^2 - y^4}}$; and conſequently $\dot{x}$ ($= \sqrt{\dot{z}^2 - \dot{y}^2}$) $= \dfrac{y^2\dot{y}}{\sqrt{a^2 - y^4}}$: Which Values, if b^2 be put $= a$ (in order to have the Powers homologous) will become $\dot{z} = \dfrac{b^2\dot{y}}{\sqrt{b^4 - y^4}}$ and $\dot{x} = \dfrac{y^2\dot{y}}{\sqrt{b^4 - y^4}}$: Whence z and x will be known. *Q. E. I.*

PROB. III.

412. *The Superficies generated by the Arch of a Curve, in its Rotation, about its Axis, being given; to determine the Curve, ſo that the Solid, itſelf, may be a* Maximum.

* Art. 145. Becauſe the Fluent of $y^2 \times \overline{\dot{z}^2 - \dot{y}^2}|^{\frac{1}{2}}$ * is to be a *Maximum*, when that of $y\dot{z}$ becomes equal to a given

Quantity; let the Fluxions here exhibited be therefore compared with $\frac{y^r \times \overline{\dot{u}\dot{u} \pm \dot{y}\dot{y}}\rvert^{n}}{\dot{y}^{2n-1}}$ and $y^m \dot{u}$ (given in the Theorem.) By means whereof (r being $= 2$, $\dot{u} = \dot{z}$, $n = \frac{1}{2}$, and $m = 1$) we have $y\dot{z} \times \overline{\dot{z}^2 - \dot{y}^2}\rvert^{-\frac{1}{2}} = a$ (a conſtant Quantity *;) which is the very Equation found in *Prob.* 1. belonging to a Circle.

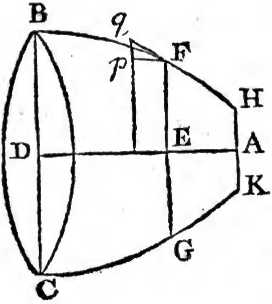
* Art. 406.

If the Solid be ſuppoſed given, and the Superficies a *Minimum*, we ſhall come at the very ſame Concluſion: For, $y^2\dot{x}$ and $y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}\rvert^{\frac{1}{2}}$ (which are reſpectively as their Fluxions) being compared with $y^m\dot{u}$ and $\frac{y^r \times \overline{\dot{u}\dot{u} + \dot{y}\dot{y}}\rvert^{n}}{\dot{y}^{2n-1}}$ we have $m = 2$, $\dot{u} = \dot{x}$, $r = 1$, and $n = \frac{1}{2}$; and therefore $\frac{\dot{x}}{y\sqrt{\dot{x}^2 + \dot{y}^2}}$ equal to a conſtant Quantity: Which being denoted by $\frac{1}{a}$ (ſo that the Terms may be homologous) there comes out $a\dot{x} = y\sqrt{\dot{x}^2 + \dot{y}^2}$; whence $2ax - x^2 = y^2$ *(as before.)*

PROB. IV.

413. *To determine the Curve* HFB, *from whoſe Revolution a Solid* BK *ſhall be generated; which, moving forward, in a Medium, in the Direction of its Axis* DA, *will be leſs reſiſted than any other Solid of the ſame given Length* DA *and Baſe* BC.

If AE $= x$, EF $= y$, F$p = \dot{x}$ *&c.* it is evident, from the Principles of Mechanics, that the reſiſting Force of a Particle of the Medium at F (being as the Square of the Sine of the Angle of Inclination pFq) will be truly repreſented by $\frac{\dot{y}\dot{y}}{\dot{x}\dot{x} + \dot{y}\dot{y}} \left(\frac{\overline{pq}\rvert^2}{\overline{Fq}\rvert^2}\right)$. Moreover, ſince



the whole Number of Particles acting upon FHKG is proportional to the Area of the Circle FG, or as y^2; the Fluxion hereof ($2y\dot{y}$) drawn into $\frac{\dot{y}\dot{y}}{\dot{x}\dot{x}+\dot{y}\dot{y}}$, will therefore give $\frac{2y\dot{y}^3}{\dot{x}\dot{x}+\dot{y}\dot{y}}$ for the Fluxion of the Reſiſtance upon FHKG.

Now, ſince it is required (by the Queſtion) to have the Fluent of $\frac{y\dot{y}^3}{\dot{x}\dot{x}+\dot{y}\dot{y}}$ (or $\frac{y\times\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-1}}{\dot{y}^{-3}}$) a *Maximum*, when That of $\dot{x}$ becomes equal to a given Quantity (AD), let theſe two Fluxions be therefore

† Art. 406. compared with $\frac{y^r\times\overline{\dot{u}\dot{u}+\dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$ and $y^m\dot{u}$ †. Whence (r being $=1$, $\dot{u}=\dot{x}$, $n=-1$, and $m=0$) we get

* Art. 406. $\frac{y\dot{x}\times\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-2}}{\dot{y}^{-3}}=a$ (a conſtant Quantity *); and conſequently $y\dot{y}^3\dot{x}=a\times\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^2$: Whereof the Fluent will be found, by *Art.* 264. That the Curve does not meet its Axis in the extreme Point A, but has an Ordinate AH at that Point (as repreſented in the Figure) is evident from the foregoing Equation. For $\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^2$, ($\overline{Fq}|^4$) being, always, greater than $\dot{y}^3\dot{x}$ ($\overline{pq}|^3\times Fp$), y muſt therefore be greater than a, in the ſame Proportion; and ſo, can never be equal to Nothing.

Now, as it is demonſtrable that the Angle AHF muſt be $\frac{3}{2}$ of a Right-Angle, AH (the leaſt Value of y) will therefore be $=4a$ (ſince $\dot{x}$ and $\dot{y}$ are, in this Circumſtance,

ſtance, equal to each other.) But, what a, itſelf, ought to be, muſt be determined from the given Values of AD and BD, and the Reſolution of the foreſaid Equation.

PROB. V.

414. *To determine the Solid of the leaſt Reſiſtance, ſuppoſing the Area of the generating Plane* AHBD, *and its greateſt Ordinate* DB *to be given; (ſee the preceding Figure.)*

Since (by the laſt Article) the Fluxion of the Reſiſtance is expreſſed by $\frac{y \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-1}}{\dot{y}^{-3}}$, and that of the Area AEFH by $y\dot{x}$, it is plain (from the premiſed Theorem *) that $\frac{\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-2}}{\dot{y}^{-3}}$ is a conſtant Quantity. * Art. 406.

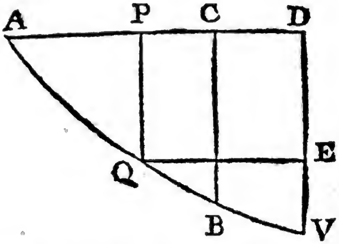
Whence, $\frac{\dot{y}^3\dot{x}}{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^2}$, or its Equal $\frac{\overline{pq}|^3 \times Fp}{\overline{qF}|^4}$, being every where the ſame, the Angle pFq muſt alſo be invariable; and conſequently HFB a Right-line. Therefore the Solid of the leaſt Reſiſtance is *(in this Caſe)* either a whole Cone, or the Fruſtrum of a, greater, Cone. But it is eaſy to ſhew, *that*, when the Area of the generating Plane AB is given ſo ſmall, that the Angle B may be taken equal to the Half of a Right-angle; I ſay, it is demonſtrable, in this Caſe, that the Fruſtrum ſo ariſing will be leſs reſiſted than a whole Cone, or any other Fruſtrum, whereof the Baſe and the Area of the generating Plane are the ſame.

In like manner the Solid of *leaſt Reſiſtance*, when its Bulk and greateſt Diameter are given, may be determined: The Equation of the generating Curve being $\frac{y^{-1}\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-2}}{\dot{y}^{-3}} = \frac{1}{a}$, or $a\dot{x}\dot{y}^3 = y \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^2$: Whereof the Solution is given in *Art.* 264.

PROB.

PROB. VI.

415. *To determine the Line, along which a Body, by its own Gravity, will, deſcend, from one given Point* A *to another* B, *in the ſhorteſt Time poſſible.*



Let AD be parallel, and BC perpendicular, to the Horizon, interſecting each other in C; and let QP be any Ordinate to the Curve parallel to BC: Then (calling AP, x; PQ, y *&c.)* the Celerity at Q will be expreſſed by $y^{\frac{1}{2}}$; alſo the Fluxion of the Time of Deſcent thro' AQ will be truly defined by $\frac{\dot{z}}{y^{\frac{1}{2}}}$ *, or its Equal $y^{-\frac{1}{2}} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{1}{2}}$. Here, therefore, the Fluent of $y^{-\frac{1}{2}} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{1}{2}}$ is to be a *Minimum*, when that of $\dot{x}$ arrives to to a given Value (AC). Whence, *by the Theorem* †, $y^{-\frac{1}{2}}\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-\frac{1}{2}}$ muſt be $=$ a conſtant Quantity: Which (to have the Terms homologous) let be denoted by $a^{-\frac{1}{2}}$ $\left(\text{or } \frac{1}{\sqrt{a}}\right)$. Then $a^{\frac{1}{2}}\dot{x} = y^{\frac{1}{2}} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{1}{2}}$; whence $\dot{x} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a-y}} = \frac{y\dot{y}}{\sqrt{ay-yy}}$; $\dot{z} = (\sqrt{\dot{x}^2+\dot{y}^2})$

* Art. 204.

† Art. 405.

$=$

$= \dfrac{a^{\frac{1}{2}}\dot{y}}{\sqrt{a-y}}$; and conſequently $z = 2a - 2a^{\frac{1}{2}}\sqrt{a-y}$.

Therefore, when $y = a$, z is $= 2a$; which two correſponding Values let be denoted by DV and AV; and let QE, parallel to AD, meet DV in E; then VE (VD — ED) being $= a - y$, and VQ (AV — AQ) $= 2a^{\frac{1}{2}}\sqrt{a-y}$, it follows that

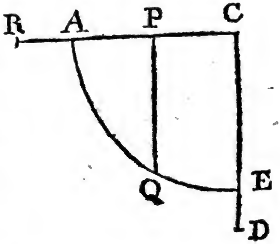
VD (a) : VE $(a-y)$ :: VA² $(4a^2)$: VQ² $(4a \times \overline{a-y})$

Which is one of the moſt remarkable Properties of the Cycloid; the Curve which, therefore, anſwers the Conditions of the Problem.

If the Celerity be ſuppoſed as any Function (S) of the Quantity y, the Problem will be reſolved in the ſame manner: The Equation of the Curve being $\dfrac{\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}^{-\frac{1}{2}}}{S} = \dfrac{1}{a}$ *.

PROB. VII.

416. *To find the Nature of the Curve* AQE, *along which a heavy Body muſt deſcend from an horizontal Line* RC *to a vertical Line* CD, *ſo that the Area* CAE *may be given, and the Time of the Deſcent a* Minimum.



If the Ordinate PQ (parallel to CD) be called y, and the Velocity at Q be denoted by y^n; it is evident that the Fluent of $y^{-n} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}^{\frac{1}{2}}$ $\left(= \dfrac{\dot{z}}{y^n}\ \dagger\right)$ muſt be a *Minimum* when that of $y\dot{x}$ amounts to a given Value.

Therefore

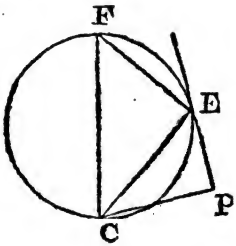
* Art. 408.

† Art. 204.

Therefore (by the Theorem already mention'd ſo often) we have $y^{-n-1}\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}^{-\frac{1}{2}} = a^{-n-1}$; and conſequently $\dot{x} = \frac{y^{n+1}\dot{y}}{\sqrt{a^{2n+2}-y^{2n+2}}}$; which, by writing $\frac{1}{2}$ inſtead of n, becomes $\dot{x} = \frac{y^{\frac{3}{2}}\dot{y}}{\sqrt{a^3-y^3}}$: Whence x will be known. But, if the Celerity was to be every where uniform, then (n being $= 0$) we ſhould have $\dot{x} = \frac{y\dot{y}}{\sqrt{a^2-y^2}}$; and therefore $x = a - \sqrt{a^2-y^2}$: Which anſwers to a Circle.

LEMMA.

417. *If, upon a Tangent* EP, *from any Point* C *in the Circumference of a Circle* FEC, *a Perpendicular* CP *be let fall, the Chord* (CE) *joining that Point and the Point of Contact, will be a Mean-Proportional between the ſaid Perpendicular* CP *and the Diameter* CF *of the Circle.*

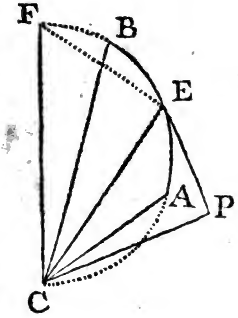


For, the Angles P and CEF being both Right; and alſo CEP = F, the Triangles CPE and CEF are ſimilar: And therefore CP : CE :: CE : CF. *Q. E. D.*

PROB. VIII.

418. *In the mixt-lin'd Triangle* ACB, *the Lengths of all the Sides (whereof* CA *and* CB *are Right-lines) are ſuppoſed given; 'tis required to find the Nature of the Curve-ſide* AEB, *ſo that the Area may be a* Maximum.

Put



Put the Arch AE $= z$, and the Ordinate CE $= y$; then, the Fluxion of the Area ACE being $\frac{y}{2}\sqrt{\dot{z}^2 - \dot{y}^2}$,* the Fluent of $y \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}^{\frac{1}{2}}$, generated in the Time wherein y, from CA, increaſes to CB, muſt be a *Maximum*: Therefore, *by the Theorem* *, we have $y\dot{z} \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}^{-\frac{1}{2}} = a$*, or $\frac{\dot{z}}{\sqrt{\dot{z}\dot{z} - \dot{y}\dot{y}}} = \frac{a}{y}$. But, if CP be ſuppoſed perpendicular to the Tangent EP, then will $\frac{\dot{z}}{\sqrt{\dot{z}\dot{z} - \dot{y}\dot{y}}}$ (*Art.* 35.) $= \frac{\text{CE}}{\text{CP}} = \frac{y}{\text{CP}}$; and conſequently $\frac{a}{y} = \frac{y}{\text{CP}}$; or, CP : CE (y) :: CE (y) : a: Which Proportion, *by the Lemma*, anſwers to a Circle; whereof the Quantity a is the Diameter.

* Art. 113.

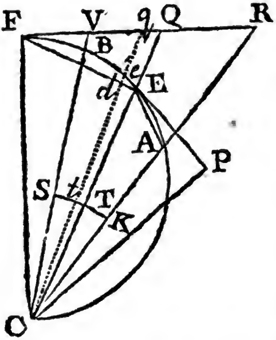
* Art. 406.

Now, that AEB muſt be an Arch of a Circle is alſo evident from *Prob.* 1. but, that the ſame Arch, continu'd out, will paſs thro' the Angle C, does not appear from thence. This is known from above; and is requiſite in finding the particular Circle anſwering to any propoſed *Data*.

PROB. IX.

419. *To find the Path* AEB *which a Body muſt deſcribe in moving uniformly from one given Point* A *to another* B; *ſo that, being every where acted on by a Force, or Virtue, which varies according to the Inverſe-Duplicate-Ratio of the Diſtances from a given Center* C, *the whole Action upon the Body ſhall be a* Minimum.

Putting



Putting AE = z, CE = y, de (indefinitely ſmall) = $\dot{y}$ *, Ee = $\dot{z}$, and Ed $(\sqrt{\dot{z}^2-\dot{y}^2})$ = $\dot{u}$; we have $\frac{\dot{z}}{y^2}$ $\left(= y^{-2}\times\overline{\dot{u}\dot{u}+\dot{y}\dot{y}}\big|^{\frac{1}{2}}\right)$ for the Meaſure of the Force which acts upon the Body in deſcribing the Particle Ee $(\dot{z})$: Moreover, if from the Center C, with any given Radius (r) an Arch KTtS of a Circle be deſcribed, interſecting CE in T, we ſhall have Tt (the Meaſure of the Angle ECe) $=\frac{r\dot{u}}{y}$. Therefore, ſince the Fluent of $y^{-2}\times\overline{\dot{u}\dot{u}+\dot{y}\dot{y}}\big|^{\frac{1}{2}}$ is required to be a *Minimum*, and the cotemporary Fluent of $y^{-1}\dot{u}$ (between CA and CB) a given Quantity; it follows, from the Theorem premiſed at the Beginning of the Section, that $y^{-2+1}\dot{u}\times\overline{\dot{u}\dot{u}+\dot{y}\dot{y}}\big|^{-\frac{1}{2}}$ muſt be equal to a conſtant Quantity $\left(\frac{1}{a}\right)$ and conſequently $\frac{\dot{u}}{\sqrt{\dot{u}\dot{u}+\dot{y}\dot{y}}}$ $\left(=\frac{\sqrt{\dot{z}^2-\dot{y}^2}}{\dot{z}}\right)=\frac{y}{a}$: Which is the very Equation found in the preceding Problem. Therefore, if thro' the three given Points A, B, and C, the Circumference of a Circle be deſcribed, the Arch thereof terminated by A and B will be the Path of the Body. *Q. E. I.*

* Art. 134.

COROLLARY.

420. If FR be a Tangent to the Circle, at the Extremity of the Diameter CF, and CA and CE be pro-

duced to meet it in R and Q, it follows that the whole Action upon the Body, in deſcribing the Arch AE, will be proportional to the correſponding Part RQ of the ſaid Tangent. For, if Ce be, alſo, produced to meet FR in q, and EF be drawn, it is plain that the Triangles CEF and CFQ, as alſo CEe and CqQ, are ſimilar: Whence it will be, CE (y) : CF (a) :: CF (a) : CQ (or Cq) $= \frac{aa}{y}$; and CE (y) : Ee $(\dot{z})$ :: Cq $\left(\frac{aa}{y}\right)$: Qq $= \frac{aa\dot{z}}{yy}$: Which (a being conſtant) is as $\left(\frac{\dot{z}}{yy}\right)$ the Force that acts upon the Body in deſcribing Ee $(\dot{z})$. And, as this every where holds, the whole Action in deſcribing AE muſt therefore be proportional to RQ. Which Force (it is eaſy to prove) will be to that exerted on the Body in moving thro' the Chord AE, as the Chord to the Arch.

PROB. X.

421. *To determine the Path in which a Body may move from one given Point* A *to another* B, *in the ſhorteſt Time poſſible; ſuppoſing the Velocity to be, every where, proportional to any Power* (y^p) *of the Diſtance from a given Center* C. (See the laſt Figure.)

Here every thing will remain as in the preceding Problem; *only* y^{-p} muſt be wrote inſtead of y^{-2}. Therefore we have $y^{-p+1}\times\dot{u}\times\overline{\dot{u}\dot{u}+\dot{y}\dot{y}}\,\Big|^{-\frac{1}{2}} =$ a conſtant Quantity: Which Quantity (to have the Terms homologous) let be denoted by $\frac{b}{a^p}$; then, by Reduction,

$$\frac{by^{p-1}}{a^p} = \frac{\dot{u}}{\sqrt{\dot{u}\dot{u}+\dot{y}\dot{y}}} \left(=\frac{\mathrm{E}d}{\mathrm{E}e}\right) = \frac{\mathrm{CP}}{\mathrm{CE}} = \frac{\mathrm{CP}}{y}:$$

And conſequently CP $= \frac{by^p}{a^p}$. Hence, if $p=0$, or the

Velocity be conſtant; then CP being every where $= b$, the Body muſt, in this Caſe, deſcribe a Right-line. But, if $p = 1$, then CP being $= \frac{by}{a}$; the Curve will be a logarithmic Spiral, whoſe Center is C *: Except in that particular Caſe, where CA = CB, when it degenerates to a Circle.

* Art. 74.

Laſtly, if $p = 2$, the Curve will be a Circle *(by the preceding Lemma)* whoſe Diameter is $\frac{aa}{b}$, and whoſe Periphery paſſes through the given Point C.

After the ſame manner, the Value of CP (upon which the Nature of the Curve depends) may be determined, when the Velocity is expounded by any given Function (S) of the Diſtance (y) from the Center of Force †: And (by writing S in the room of $y^n +$ *&c.*) will come out CP $= \frac{bS}{c}$; where b and c repreſent conſtant Quantities.

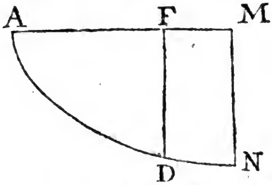
† Art. 407.

When the Velocity is That which the Body may acquire, in deſcending thro' BE, by a centripetal Force expreſſed by y^p, then the Value of S (the Meaſure of that Velocity) being interpreted by $\sqrt{d^{p+1} - y^{p+1}}$ ‡ (where CB $= d$) we therefore have CP $= \frac{b\sqrt{d^{p+1} - y^{p+1}}}{c}$ for the Equation of the Curve of the ſwifteſt Deſcent, according to this laſt *Hypotheſis* of a centripetal Force varying as any Power p of the Diſtance.

‡ Art. 221. and 206.

422. Beſides the Problems already reſolved in this Section, there are others of the ſame Nature which are confined to more particular Reſtrictions, and require a different Method of Solution.

Thus, if Q, R and S be ſuppoſed to denote any given Powers, or Functions, of the Ordinate (y) of a Curve ANM, and the Nature of the Curve be required, ſo that, when the Fluent of $Q\dot{x}$ becomes equal to a given Quantity, the Fluent of $R\dot{z}$ may alſo become equal to another given Quantity, and That of $S\dot{z}$, a *Maximum* or *Minimum*: Then, becauſe there is, in this Caſe, a ſecond Equation, or new Condition, beyond what is to be met with in any of the foregoing Problems, the Method of Solution hitherto explained, will, therefore, be inſufficient. But; by a Proceſs ſimilar to that whereby the ſaid Method was demonſtrated (aſſuming, here, three Expreſſions, and three indeterminate Quantities, inſtead of two *) a general Anſwer to this Problem (under all its Reſtrictions) will be obtained: And is exhibited by the Equation, $\frac{\dot{z}}{\dot{x}} = \frac{pR \pm qS}{Q}$; wherein p and q denote conſtant Quantities.

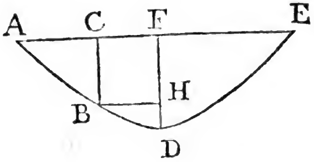


* Art. 407.

423. Though it ſeems unneceſſary to put down the Invention of this Equation, after what has been hinted above, yet it may not be improper to obſerve, by way of Corollary, that, if $Q=1$, $R=1$, and $S=y^n$, the Equation will then become $\frac{\dot{z}}{\dot{x}} = p \pm qy^n$; expreſſing the Nature of the Curve, when, the whole Abſciſſa (AM) and correſponding Arch (AN) being both given Quantities, the Fluent of $y^n\dot{z}$ is a *Maximum* or *Minimum*, according as the Value of n is poſitive or negative: In both which Caſes, it is very eaſy to perceive, that the Curve muſt be concave to AM, and that the Value of $\frac{\dot{z}}{\dot{x}}$, or its

Equal $p \pm qy^n$, muſt, therefore, decreaſe as y increaſes whence we may infer that the Sign of qy^n muſt be negative in the former Caſe, and poſitive in the latter.

Ex. Let the Curve ABDE, be the *Catenaria*; form'd by a ſlender Chain, or perfectly flexible Cord,



ſuſpended by its two Extremes in the horizontal Line AE: Then, ſince its Center of Gravity muſt be the loweſt poſſible, the Fluent of $y\dot{z}$, when AC=AE, muſt † Art. 173. therefore be a *Maximum* †: Whence (n being here $=1$) our Equation $\left(\frac{\dot{z}}{\dot{x}} = p \pm qy^n\right)$ becomes $\frac{\dot{z}}{\dot{x}} = p - qy$.

But, in order to reduce it to a more convenient Form, let the Diſtance (DF) of the loweſt Point of the Curve from the horizontal-Line AE be put $= b$; then, when y (BC) becomes $= b$, $\dot{x}$ will be $= \dot{z}$; and therefore the Equation, in that Circumſtance, is $1 = p - qb$; whence $p = 1 + qb$, and conſequently $\frac{\dot{z}}{\dot{x}} = 1 + qb - qy = 1 + q \times \overline{b-y}$: Which, by putting $b - y$ (DH) $= s$ and $a = \frac{1}{q}$ is reduced to $\frac{\dot{z}}{\dot{x}} = 1 + \frac{s}{a}$: From whence $a^2\dot{z}^2$ $(= \overline{a+s}^2 \times \dot{x}^2) = \overline{a+s}^2 \times \overline{\dot{z}^2 - \dot{s}^2}$; and conſequently BD $= \sqrt{2as + ss}$.

For another Example (wherein the Exponent n will be negative) let the required Curve be That along which a Body may deſcend, by its own Gravity, from one given Point A to another B, in leſs Time than thro' any other Line of the ſame Length. In which Caſe, the Fluent of $\dot{z}y^{-\frac{1}{2}}$ being a *Minimum*, when x and z become equal to given Quantities, our Equation (by writing $-\frac{1}{2}$ for n) will here become $\frac{\dot{z}}{\dot{x}} = p + qy^{-\frac{1}{2}}$: From whence exterminating x, or $\dot{z}$, by means of the Equation $\dot{x}^2 + \dot{y}^2 = \dot{z}^2$, the Fluent may alſo be determined.

SECTION XI.

The Reſolution of Problems of various Kinds.

PROB. I.

424. *ANY hyperbolical Logarithm* (y) *being given, 'tis propoſed to find the natural Number anſwering thereto.*

If the Number ſought be denoted by $1+x$, we ſhall (*by Art.* 126.) have $\dot{y} = \frac{\dot{x}}{1+x}$, or $\dot{y} + x\dot{y} - \dot{x} = 0$. Let $Ay + By^2 + Cy^3$ &c. $= x$; then $A\dot{y} + 2By\dot{y} + 3Cy^2\dot{y}$ &c. $= \dot{x}$, and our Equation will become

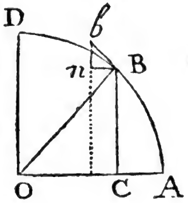
$$\left.\begin{matrix} \dot{y} + Ay\dot{y} + By^2\dot{y} + Cy^3\dot{y} \text{ \&c.} \\ - A\dot{y} - 2By\dot{y} - 3Cy^2\dot{y} - 4Dy^3\dot{y} \text{ \&c.} \end{matrix}\right\} = 0.$$

Whence, by comparing the homologous Terms, we get $A = 1$, $B = \frac{A}{2} = \frac{1}{2}$, $C = \frac{B}{3} = \frac{1}{2 \cdot 3}$, $D = \frac{C}{4} = \frac{1}{2 \cdot 3 \cdot 4}$ &c. Therefore $1 + y + \frac{y^2}{2} + \frac{y^3}{2 \cdot 3} + \frac{y^4}{2 \cdot 3 \cdot 4} + \frac{y^5}{2 \cdot 3 \cdot 4 \cdot 5}$ &c. is $(= 1 + x)$ the Number ſought.

PROB. II.

425. *The Radius* AO *and any Arch* AB *of a Circle* ABD *being given; to find the Sine* BC, *and Co-ſine* OC *of that Arch.*

Let AO (BO) $= r$, AB $= z$, AC $= x$, BC $= y$,



B$b = \dot{z}$, B$n = \dot{x}$, and $bn = \dot{y}$: Becauſe of the ſimilar Triangles OBC and Bnb, it will be

$$\text{OB } (r) : \text{BC } (y) :: \text{B}b\ (\dot{z}) : \text{B}n\ (\dot{x})$$

$$\text{And OB } (r) : \text{OC } (r-x) :: \text{B}b\ (\dot{z}) : bn\ (\dot{y})$$

From which we have

$$y\dot{z} = r\dot{x}$$

$$\text{And } r\dot{y} = r\dot{z} - x\dot{z}.$$

Let $x = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5$ &c.
And $y = az + bz^2 + cz^3 + dz^4 + ez^5$ &c.

Then, by Subſtitution and Tranſpoſition, our two Equations will become

$$\left.\begin{array}{l} * \quad + az\dot{z} + bz^2\dot{z} + cz^3\dot{z} + dz^4\dot{z} \text{ \&c.} \\ -rA\dot{z} - 2rBz\dot{z} - 3rCz^2\dot{z} - 4rDz^3\dot{z} - 5rEz^4\dot{z} \text{ \&c.} \end{array}\right\} = 0$$

And

$$\left.\begin{array}{l} ra\dot{z} + 2rbz\dot{z} + 3rcz^2\dot{z} + 4rdz^3\dot{z} + 5rez^4\dot{z} \text{ \&c.} \\ -r\dot{z} + Az\dot{z} + Bz^2\dot{z} + Cz^3\dot{z} + Dz^4\dot{z} \text{ \&c.} \end{array}\right\} = 0$$

From which, by equating the homologous Terms, we get $A = 0$, $a = 2rB$, $b = 3rC$, $b = 4rD$, $d = 5rE$ &c.

Alſo $a = 1$, $b = -\dfrac{A}{2r}$, $c = -\dfrac{B}{3r}$, $d = -\dfrac{C}{4r}$ &c.

Therefore $2rB = 1$, $3rC = -\frac{A}{2r}$, $4rD = -\frac{B}{3r}$, $5rE = -\frac{C}{4r}$, &c. and conſequently $B = \frac{1}{2r}$, $C = 0$, $D = -\frac{B}{3\cdot 4r^2} = -\frac{1}{2\cdot 3\cdot 4r^3}$, $E = 0$, $F = -\frac{D}{5\cdot 6r^2} = \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot r^5}$ &c.

Whence, alſo b $(= 3rC) = 0$, c $(= 4rD) = -\frac{1}{2\cdot 3r^2}$ &c. &c.

Hence it is evident that y $(= az + bz^2 + cz^3$ &c.) $= z - \frac{z^3}{2\cdot 3r^2} + \frac{z^5}{2\cdot 3\cdot 4\cdot 5r^5} - \frac{z^7}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7r^6}$ + &c. And that x $(= Az + Bz^2 + Cz^3$ &c.) $= \frac{z^2}{2r} - \frac{z^4}{2\cdot 3\cdot 4r^3} + \frac{z^6}{2\cdot 3\cdot 4\cdot 5\cdot 6r^5}$ — &c. ‡

PROB. III.

426. *To find the Value of* x, *when* x^x *is a* Minimum.

The Logarithm of x^x is $= x \times l.x$; whoſe Fluxion $\dot{x} \times l.x + \dot{x}$ being $= 0$, we have $l:x = -1$. But (*by Prob.* 1.) the Number whoſe hyp. Log. is y will be $1 + y + \frac{y^2}{2} + \frac{y^3}{2\cdot 3} + \frac{y^4}{2\cdot 3\cdot 4}$ &c. Therefore, by writing -1 inſtead of y, we have $x = 1 - 1 +$

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‡ *The Subſtance of this Solution (being the moſt neat and artful I have ſeen to that uſeful Problem) I had from a Letter ſign'd* —— Needler; *which was put into my Hands by a Friend, who receiv'd it from the late Dr.* Halley, *to whom it was wrote.*

$\frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}$ &c. $= 0{,}367878$ &c.

PROB. IV.

427. *To divide a given Number (a) ſo that the continual Product of all its Parts may be a* Maximum.

It is evident (*from Art.* 23.) that all the Parts muſt be equal: If, therefore, any one of them be denoted by x, their Number will be $\frac{a}{x}$, and we ſhall have $x^{\frac{a}{x}}$ a *Maximum:* And therefore its Logarithm $\frac{a}{x} \times L.x$ a *Maximum* alſo: And its Fluxion $-\frac{a\dot{x}}{x^2} \times L.x$

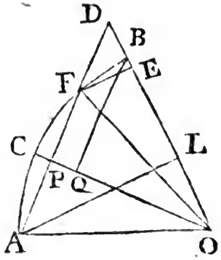
* Art. 22. and 126. $-\frac{a\dot{x}}{x^2} = 0$ *: Whence H-$L.x = 1$, and conſequently

† Art. 424. $x = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4}$ &c. $= 2.71828$ &c. Therefore the next inferior, or ſuperior, Number to 2,71828 &c. that will exactly meaſure the given Number a, is the required Value of each Part: Thus, let $a = 10$; then becauſe $\frac{10}{2.71828 \text{ \&c.}} = 4$ nearly, the Number of Parts, in this Caſe, will be 4, and the Value of each $= \frac{10}{4} = 2.5$.

PROB. V.

428. *To divide a given Angle* AOB *into two Parts* AOC *and* BOC, *ſo that the Product of any given Powers,* $AP^n \times BQ^m$, *of their Sines* AP *and* BQ *may be a* Maximum.

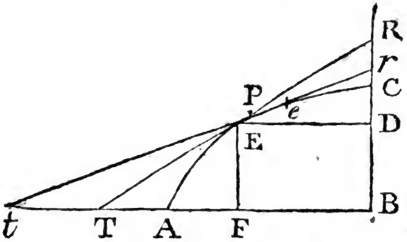
Let AP, produced, cut the Radius OB in D, and the Arch AB in F; likewiſe let FE and AL be perpendicular to OB, and join O, F: Putting $AO = r$, $AP = x$ and $BQ = y$. Then, becauſe $x^n y^m$ is to be a *Maximum*, we have $n x^{n-1}\dot{x} \times y^m + x^n \times m y^{m-1}\dot{y} = 0$; and conſequently $ny\dot{x} = -mx\dot{y}$.



Moreover, ſince the Fluxion of the Arch AC is $= \frac{r\dot{x}}{\sqrt{r^2-x^2}}$ and that of BC $= \frac{r\dot{y}}{\sqrt{r^2-y^2}}$ (*Art.* 142.) we alſo have $\frac{r\dot{y}}{\sqrt{r^2-y^2}} + \frac{r\dot{x}}{\sqrt{r^2-x^2}} = 0$, or $\frac{\dot{y}}{\sqrt{r^2-y^2}} = \frac{-\dot{x}}{\sqrt{r^2-x^2}}$; which multiply'd by the former Equation, &c. gives $\frac{ny}{\sqrt{r^2-y^2}} = \frac{mx}{\sqrt{r^2-x^2}}$, or $n \times \frac{y\sqrt{r^2-x^2}}{\sqrt{r^2-y^2}} = mx$: Whence, becauſe OQ $(\sqrt{r^2-y^2})$: QB (y) :: OP $(\sqrt{r^2-x^2})$: PD $= \frac{y\sqrt{r^2-x^2}}{\sqrt{r^2-y^2}}$, we have $n \times PD\ (= mx) = m \times AP$; and therefore PD : AP :: $m : n$; whence (by Compoſition and Diviſion) AD : DF :: $m+n : m-n$: But (*by ſim. Triang.*) AD : DF :: AL : FE; conſequently $m+n : m-n$ :: AL : FE; that is, as the Sum of the Indices of the two propoſed Powers is to their Difference, ſo the Sine of the whole given Angle to the Sine of the Difference of its two, required, Parts. This Proportion is given in Words, at length, becauſe it will be found of frequent Uſe in the Solution of mechanical Problems.

PROB. VI.

429. *To ſhew that the leaſt Triangle that can be deſcribed about, and the greateſt Parallelogram in, a given Curve* ABC, *concave to its Axis, will be when the Subtangent* FT *is equal to the Baſe* BF *of the Parallelogram, or half the Baſe* BT *of the Triangle.*



It appears from *Art.* 25. and is demonſtrable by common Geometry, that the greateſt Parallelogram that can be inſcrib'd in the Triangle BTR (ſuppoſing the Poſition of TR to remain the ſame) will be *that* whoſe Baſe BF is half the Baſe BT of the Triangle: Therefore, as a greater Figure cannot poſſibly be inſcribed in the Curve BAC than in the Triangle BTR circumſcribing it, the greateſt Parallelogram that can be inſcribed, either, in the Triangle or the Curve, muſt be That above ſpecified.

But now, to make it alſo appear that the Triangle BTR is a *Minimum* when FT = BT; let B*tr* be any other circumſcribing Triangle, and let the two Tangents TER and *ter* interſect each other in P. Then, ER being = ET, it is plain that RP is leſs than PT, and P*r* (leſs than PR leſs than PT) leſs than P*t*: Therefore, the Sides PR and P*r* of the Triangle RP*r* being leſs than the Sides, PT and P*t* of the Triangle TP*t*, and the oppoſite Angles RP*r* and TP*t* equal to each other, it follows that the Triangle RP*r* is leſs than TP*t*; and conſequently, by adding the Trapezium BTP*r* to both, it appears that BTR is leſs than B*tr*.

Co-

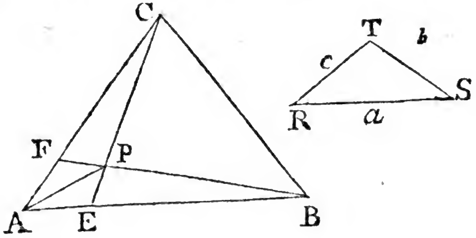
COROLLARY.

430. Hence the greateſt inſcribed Parallelogram is half the leaſt circumſcribing Triangle.

In the ſame Way it may be proved, that the greateſt inſcrib'd Cylinder, and the leaſt circumſcribing Cone, in, and about, the Solid generated by Revolution of a given Curve, will be when the Sub-tangent is equal to twice the Altitude of the Cylinder, or $\frac{2}{3}$ of the Altitude of the Cone: And that the two Figures will be to each other in the Ratio of 4 to 9.

PROB. VII.

431. *Three Points* A, B, C *being given, to find the Poſition of a fourth Point* P, *ſo that, if Lines be drawn from thence to the three former, the Sum of the Products* $a\times AP$, $b\times BP$, *and* $c\times CP$ *(where a, b and c denote given Numbers) ſhall be a* Minimum.

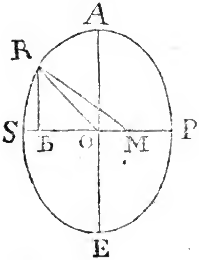


If CP and BP be produced to E and F, it will appear from *Art.* 35. and 36. that the Sine of BPE muſt be to that of APE, as *a* to *b*; and the Sine of CPF (BPE) to that of APF, as *a* to *c*. Therefore, the Sines of the three Angles BPE, APE, and APF (which Angles, taken all together, make two Right-ones) being in the given Ratio of *a*, *b* and *c*, it follows, that, if a Triangle RST be conſtructed, whoſe Sides RS, ST and RT are in the ſaid Ratio of *a*, *b* and *c*, the Angles T, R and S oppoſite thereto, will be reſpectively equal to the fore-mention'd Angles BPE, APE and APF. From whence, all the Angles at the Point P being given, the Poſition of that Point is given by common Geometry.

But it is obſervable, that, when one of the three given Quantities a, b, c (ſuppoſe a) is equal to, or greater than, the Sum of the other two, a Triangle cannot then be formed whoſe Sides are proportional to the ſaid Quantities : In that Caſe the Point P will fall in the Point (A) correſponding to the greateſt Quantity (a). For, it is plain that $b\times AB$ is leſs than $b\times BP + b\times AP$; and that $c\times AC$ is leſs than $c\times CP + c\times AP$; whence, by adding the Leſs to the Leſs, and the Greater to the Greater, it alſo appears that $b\times AB + c\times AC$ muſt be leſs than $b\times BP + c\times CP + \overline{b+c}\times AP$ leſs than $b\times BP + c\times CP + a\times AP$; becauſe a (by Hypotheſis) is equal to, or greater than, $b+c$.

PROB. VIII.

432. *To determine in what Latitude a Right-line perpendicular to the Surface of the Earth, and Another drawn, from the ſame Point, to the Center, make the greateſt Angle, poſſible, with each other; the Ratio of the Axis and the Equatoreal Diameter being ſuppoſed given.*



Let AE repreſent the Equatoreal Diameter, and SP the Axis of the Earth (taken as an oblate Spheroid) alſo let RO and RM repreſent the two Lines ſpecified in the Problem, whereof let the latter (perpendicular to ARS) meet SP in M; and let RB be perpendicular to SP.

It is evident, from the Property of the Ellipſis, that $SP^2 : AE^2 :: BO : BM$. And (by Trigonometry) BO : BM :: Tang. BRO : Tang. BRM; whence, by Equality,

lity, SP^2 : AE^2 :: Tang. BRO : Tang. BRM; therefore, by Compoſition and Diviſion, AE^2+SP^2 : AE^2—SP^2 :: Tang. BRM+Tang. BRO : Tang. BRM—Tang. BRO. But, *the Sum of the Tangents of any two Angles is to their Difference, as the Sine of the Sum of thoſe Angles to the Sine of their Difference* *; whence it follows that AE^2 + SP^2 : AE^2—SP^2 :: Sine. $\overline{\mathrm{BRM}+\mathrm{BRO}}$: Sine. $\overline{\mathrm{BRM}-\mathrm{BRO}}$ (ORM).

Now, ſince the Ratio of the two firſt Terms is conſtant, or in every Part of the Ellipſis the ſame, it is obvious that the Angle ORM, or its Sine, will be the greateſt poſſible, when its Antecedent (the Sine of $\overline{\mathrm{BRM}+\mathrm{BRO}}$) is the greateſt poſſible, that is when BRM+BRO = a Right-Angle and its Sine = Radius. Therefore, in the propoſed Circumſtance, when ORM is a *Maximum*, our laſt Proportion will become AE^2+SP^2 : AE^2—SP^2 :: Radius : Sine of ORM: And half the Angle, ſo found, added 45°, will give (BRM) the Complement of the required Latitude; becauſe BRM + BRO (or 2BRM—ORM) being = 90°, it is evident that 2BRM=90+ORM, and conſequently BRM = 45° + $\frac{1}{2}$ ORM.

PROB. IX.

433. *Of all the Semi-cubical Parabolas, to determine that, whereof, the Length of the Curve being given, the Area ſhall be a* Maximum.

The general Equation is $ax^2 = y^3$: Moreover, the Area is univerſally $= \frac{3y^{\frac{5}{2}}}{5a^{\frac{1}{2}}}$, and the Length of the Curve $= \frac{\overline{4a+9y}^{\frac{3}{2}}}{27a^{\frac{1}{2}}} - \frac{8a}{27}$ (*ſee Art.* 137.) Let the laſt of theſe be put = c, and, by ordering the Equation, you will get $y = \frac{a^{\frac{1}{3}} \times \overline{27c + 8a}|^{\frac{2}{2}} - 4a}{9}$: Whence, $\frac{3y^{\frac{5}{2}}}{5a^{\frac{1}{2}}}$ (and conſequently $\frac{y}{a^{\frac{1}{5}}}$) being a *Maximum*, it is evident that $\frac{a^{\frac{1}{3}} \times \overline{27c + 8a}|^{\frac{2}{3}} - 4a}{a^{\frac{1}{5}}}$, or its Equal $a^{\frac{2}{15}} \times \overline{27c + 8a}|^{\frac{2}{3}} - 4a^{\frac{4}{5}}$ muſt likewiſe be a *Maximum:* Which, put into Fluxions and reduced, gives $a = c \times \frac{9 + 3\sqrt{21}}{32}$: Whence x and y will alſo be found.

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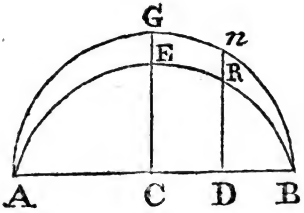
* *Vid. p.* 55. *of my Trigonometry.*

PROB. X.

434. *To determine the Ratio of the Periphery of any given Ellipſis to that of its circumſcribing Circle.*

Call the Semi-tranſverſe Axis CB, a; the Semi-conjugate CE, c; any Ordinate DR, y; and its Diſtance



CD from the Center, x: Then (by the Nature of the Curve) y being $= \frac{c}{a}\sqrt{aa - xx}$, we have $\dot{y} = \frac{-cx\dot{x}}{a\sqrt{aa - xx}}$; and conſequently $\dot{z}$ $(\sqrt{\dot{x}^2 + \dot{y}^2}) = \frac{\dot{x}\sqrt{a^4 - \overline{a^2 - c^2} \times x^2}}{a\sqrt{aa - xx}}$: Which, by making $d =$

$\frac{aa - cc}{aa}$ will be reduced to $\dot{z} = \frac{\dot{x}\sqrt{aa - dxx}}{\sqrt{aa - xx}} =$

$$\frac{a\dot{x}}{\sqrt{aa - xx}} \times \overline{1 - \frac{dx^2}{2a^2} - \frac{d^2x^4}{2 \cdot 4a^4} - \frac{3d^3x^6}{2 \cdot 4 \cdot 6a^6}\ \&c.}$$

(by throwing the Numerator into a Series) whereof the *whole* Fluent, when x becomes $= a$, will be z (ERB)

$$= A \times \overline{1 - \frac{d}{2 \cdot 2} - \frac{3d^2}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{3 \cdot 3 \cdot 5d^3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7d^4}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8}}\ \&c.$$

(*by Art.* 286.) where A denotes the Length of the Arch G*n*B, or $\frac{1}{4}$ of the Periphery of the circumſcribing Circle.

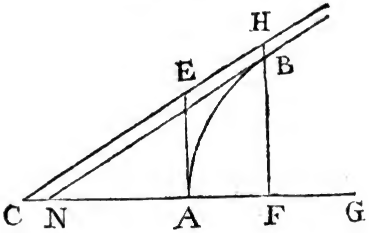
Hence it follows that the Periphery of the Ellipſis is to that of its circumſcribing Circle, as $1 - \frac{d}{2 \cdot 2} - \frac{3d^2}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{3 \cdot 3 \cdot 5d^3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}$ &c. or as $1 - \frac{d}{2 \cdot 2} \times A + \frac{1 \cdot 3d}{4 \cdot 4} \times B + \frac{3 \cdot 5d}{6 \cdot 6} \times C + \frac{5 \cdot 7d}{8 \cdot 8} \times D$ &c. to Unity: Where A, B, C, D &c. denote the preceding Terms, under their proper Signs.

PROB. XI.

435. *To determine the Difference between the Length of the Arch of a Semi-hyperbola infinitely produced, and its Aſymptote.*

Call the Semi-tranſverſe Axis (AC) a; the Semi-conjugate (or its Equal AE) b; the Diſtance (CF) of any Ordinate from the Center, x; the Ordinate itſelf, y; and the Arch correſponding, z: Then, from the Nature of the Curve we have $y = \frac{b\sqrt{x^2 - a^2}}{a}$; whence

$\dot{y} =$



$\dot{y} = \frac{bx\dot{x}}{a\sqrt{x^2 - a^2}}$; and conſequently $\dot{z}\left(=\sqrt{\dot{x}^2+\dot{y}^2}\right) = \frac{\dot{x}\sqrt{\frac{aaxx+bbxx}{aa} - a^2}}{\sqrt{xx-aa}}$: Which, making $d^2 = \frac{a^2}{a^2+b^2}$ $\left(= \frac{\text{CA}^2}{\text{CE}^2}\right)$ and $u = \frac{a}{x}$ will be transformed to $\dot{z} = -\frac{a\dot{u}}{du^2} \times \frac{\overline{1-dduu}|^{\frac{1}{2}}}{\overline{1-uu}|^{\frac{1}{2}}}$; whereof the upper Surd, expanded, is $= 1 - \frac{d^2u^2}{2} - \frac{d^4u^4}{8}$ &c. And therefore $\dot{z} = \frac{a}{d}$ into $\frac{-\dot{u}}{u^2\sqrt{1-uu}} + \frac{d^2\dot{u}}{2\sqrt{1-uu}} + \frac{d^4u^2\dot{u}}{8\sqrt{1-uu}} + \frac{3d^6u^4\dot{u}}{8.6\sqrt{1-uu}} + \frac{3.5\,d^8u^6\dot{u}}{8.6.8\sqrt{1-uu}}$ &c. Now the Fluent of the firſt Term hereof, $\frac{a}{d}$ into $\frac{-\dot{u}}{u^2\sqrt{1-uu}}$ $\left(= \frac{x\dot{x}}{d\sqrt{x^2-a^2}}\right)$ is univerſally expreſſed by $\frac{\sqrt{x^2-a^2}}{d}$, or its Equal $\frac{\text{BF} \times \text{CE}}{\text{AE}}$: Which, if BN be parallel to the Aſymptote EC, will (becauſe AE : CE

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CE :: BF : BN) be alſo truly repreſented by BN : And this Line BN, when x or z becomes infinite, will coincide with the Aſymptote. Therefore the Fluent of the remaining Terms is the Difference ſought: Which Fluent, when $u = 1$, or $y = 0$ (putting A for $\frac{1}{4}$ of the Periphery of the Circle whoſe Radius is Unity) will be

$$= aA \times \overline{\frac{d}{2} + \frac{d^3}{2 \cdot 2 \cdot 4} + \frac{3 \cdot 3d^5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6.} + \frac{3 \cdot 3 \cdot 5 \cdot 5d^7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8} + \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7d^9}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10}} \text{ \&c.}$$

(*by Art.* 286.) but $= 0$ when $u = 0$ (or y is infinite). Therefore the Exceſs of the Aſymptote above the Curve is truly exhibited by the preceding Series. *Q. E. I.*

If a be taken $= 1$, and $b = 0$, then d will become $= 1$: And therefore, the Curve in this Caſe falling into its Axis AG, we have

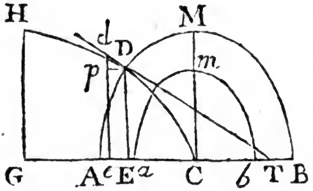
$$A \times \overline{\frac{1}{2} + \frac{1}{2 \cdot 2 \cdot 4} + \frac{3 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} + \frac{3 \cdot 3 \cdot 5 \cdot 5.}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8}} \text{ \&c.} = \text{CA},$$

or Unity. Whence it appears that the Sum of the Series $\frac{1}{2} + \frac{1}{2 \cdot 2 \cdot 4} + \frac{3 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6}$ is the Reciprocal of $\frac{1}{4}$ of the Periphery of the Circle whoſe Radius is Unity. And, from the Problem preceding the laſt, it will likewiſe appear, that the Sum of the Series $1 - \frac{1}{2 \cdot 2} - \frac{3}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{3 \cdot 3 \cdot 5 \cdot}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}$ &c. will be denoted by the ſame Quantity; and conſequently that theſe two Serieſes are equal to each other. From the Addition and Subtraction of which and their Multiples, various other Serieſes may be produced, whoſe Sums are explicable by means of the Periphery of a Circle.

PROB.

PROB. XII.

436. *To determine the Nature of the Curve* CDH, *which will interſect any Number of ſimilar and concentric Ellipſes* AMB, *amb* &c. *at Right-Angles.*



Let the Tangent DT, which is a Normal to the Ellipſis AMB, meet the Axis AB in T; and, ſuppoſing AC, CM, aC, Cm *&c.* to be the principal Semi-diameters of their reſpective Ellipſes, let the given Ratio of AC^2 to CM^2 (or of aC^2 to Cm^2 *&c.*) be that of 1 to n: Putting $CE = x$, $ED = y$, Dp $(Ee) = \dot{x}$, and $dp = \dot{y}$.

It is a known Property of the Ellipſis that $AC^2 : CM^2 :: CE : ET$; therefore $ET = nx$: Moreover ET $(nx) : Dp$ $(\dot{x}) :: ED$ $(y) : pd$ $(\dot{y})$ by ſimilar Triangles) whence $\frac{\dot{x}}{nx} = \frac{\dot{y}}{y}$, or $\frac{\dot{x}}{x} = \frac{n\dot{y}}{y}$; whereof the Fluent

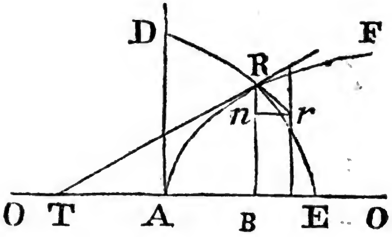
* Art 126. is $L : x - L : a = nL : y - nL : a$ * (where a denotes any conſtant Quantity at Pleaſure.) Hence we alſo have $L : \frac{x}{a} = n \times L : \frac{y}{a} = L : \frac{y^n}{a^n}$, and conſequently

$$\frac{x}{a} = \frac{y^n}{a^n}, \text{ or } a^{n-1}x = y^n.$$

PROB. XIII.

437. *To find the Equation of a Curve* ERD *that will cut any Number of Ellipſes, or Hyperbolas, having the ſame Center* O *and Vertex* A, *at Right-Angles.*

Let RT be a Tangent to any one of the propoſed Conic Sections ARF, at the Interſection R, meeting the



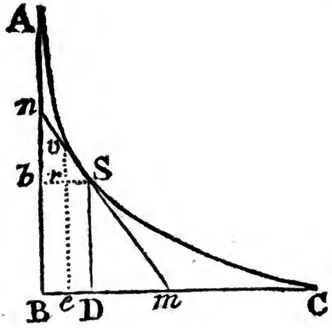
the Axis AO in T; and put AO$=a$, OB$=x$, BR$=y$, $nr=\dot{x}$, R$n=-\dot{y}$: Then *(per Conics)* BT $=\frac{a^2-x^2}{x}$, in the Ellipſis, and $=\frac{x^2-a^2}{a^2}$, in the Hyperbola: Whence, by reaſon of the ſimilar Triangles TBR, and R*rn*, it will be $\frac{a^2 \backsim x^2}{x}$ (BT) : y (BR) :: $-\dot{y}$ (R*n*) : $\dot{x}$ (*rn*): Therefore $-y\dot{y}=\frac{a^2\dot{x} \backsim x^2\dot{x}}{x}=\frac{a^2\dot{x}}{x} \backsim x\dot{x}$, and conſequently $-\frac{y^2}{2}+d^2=a^2 \times \mathrm{L} : \frac{x}{a} \backsim \frac{1}{2}x^2$. Where d denotes a conſtant Quantity, depending on the given Value of AE.

PROB. XIV.

438. *Let two Points n and m move, at the ſame time, from two given Poſitions* B *and* C, *with equal Celerities, along two Right-lines* BA *and* BC *perpendicular to each other: 'Tis propoſed to determine the Curve* ASC, *to which a Right-line joining the ſaid Points ſhall, always, be a Tangent.*

Let DS and *ev* be parallel to BA, and S*rb* perpendicular thereto: Putting BC$=a$, CD$=x$, SD$=y$, S*r* $=\dot{x}$, and $rv=\dot{y}$. Therefore *(by ſim. Triangles)* $\dot{y}:\dot{x}$ :: y

:: $y : \frac{y\dot{x}}{\dot{y}} = \mathrm{D}m$, and $\dot{x} : \dot{y} :: a - x$ (S*b*) : $\frac{\overline{a-x}\times\dot{y}}{\dot{x}} =$



bn: Whence C*m* (CD—D*m*) $= x - \frac{y\dot{x}}{\dot{y}}$, and B*n* (B*b* + *bn*) $= y + \frac{\overline{a-x}\times\dot{y}}{\dot{x}}$: Which two laſt Values, becauſe the Velocities of the Bodies are equal, muſt alſo be equal to each other, that is, $x - \frac{y\dot{x}}{\dot{y}} = y + \frac{\overline{a-x}\times\dot{y}}{\dot{x}}$: Hence, by making $\dot{x}$ conſtant, and taking the Fluxion of the whole Equation, we get $\dot{x} - \frac{\dot{y}\dot{x}\dot{y}+y\dot{x}\ddot{y}}{\dot{y}^2} = \dot{y} - \frac{\dot{x}\dot{y}+\overline{a-x}\times\ddot{y}}{\dot{x}}$; or $\frac{\overline{a-x}\times\ddot{y}}{\dot{x}} = \frac{y\dot{x}\ddot{y}}{\dot{y}^2}$; from which there ariſes $\overline{a-x}\times\dot{y}^2 = y\dot{x}^2$, and $\frac{\dot{y}}{\sqrt{y}} = \frac{\dot{x}}{\sqrt{a-x}}$: Where, the Fluent on both Sides being taken, we have $2\sqrt{y} = 2\sqrt{a} - 2\sqrt{a-x}$, and conſequently $x = 2\sqrt{ay} - y$: Which Equation pertains to the common Parabola.

Otherwiſe more univerſally, thus.

439. Put Cm=v and Bn=w, and let theſe Quantities (inſtead of being equal) have any given Relation to each other. Then, ſince the abſolute Celerity of m is expreſſed by $\dot{v}$, its angular Celerity, in a Direction perpendicular to Sm, by which the Line Sm tends to revolve about the Point of Contact S as a Center, will be truly defined by $\frac{\textit{Sine of Bmn}}{\textit{Radius}} \times \dot{v}$ (*Art.* 35.) In the ſame manner the angular Celerity of n, about the Point S, will be defined by $\frac{\textit{Sin. Bnm}}{\textit{Rad.}} \times \dot{w}$. Now, as theſe Celerities muſt be to each other as the Diſtances Sm and Sn from the Center S (or directly as the Radii) we have Sm : Sn (:: DS : bn) :: *Sin. Bmn* × $\dot{v}$: *Sin. Bnm* × $\dot{w}$; whence, becauſe *Sin. Bmn* : *Sin. Bnm* :: *Bn* (w) : *Bm* $(a-v)$ we alſo have DS : bn :: $w \times \dot{v} : \overline{a-v} \times \dot{w}$: Therefore, by Compoſition, DS : (DS + bn) w :: $w\dot{v} : w\dot{v} + \overline{a-v} \times \dot{w}$, and conſequently DS $= \frac{w^2\dot{v}}{w\dot{v} + \overline{a-v} \times \dot{w}}$: Whence bn (w — SD) $= \frac{\overline{a-v} \times w\dot{w}}{w\dot{v} + \overline{a-v} \times \dot{w}}$; and BD $\left(= \mathrm{S}b = \frac{bn \times Bm}{Bn}\right)$ $= \frac{\overline{a-v}|^2 \times \dot{w}}{w\dot{v} + \overline{a-v} \times \dot{w}}$: From whence the Curve itſelf will be given.

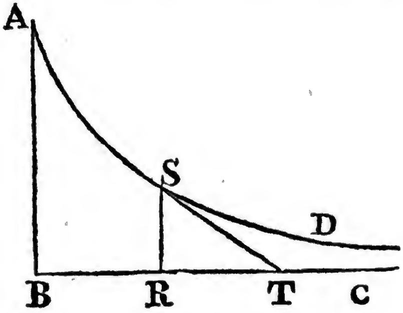
If v and w be taken equal to each other (as above) then SD (y) will become $= \frac{w^2}{a}$, and BD $= \frac{\overline{a-w}|^2}{a}$ $= a - 2w + \frac{w^2}{a}$; in which laſt, if for w its Equal

$\sqrt{ay}$ be ſubſtituted, we ſhall have BD $= a - 2\sqrt{ay} + y$; and conſequently CD $(a - \text{BD}) = 2\sqrt{ay} - y$, the very ſame as before.

PROB. XV.

440. *Suppoſing a Body* T *to proceed, uniformly, along a Right-line* BC, *and another Body* S, *in purſuit of the ſame, always directly towards it, with a Celerity which is to that of* T, *in any given Ratio, of* 1 *to* n; *'tis propoſed to find the Equation of the Curve* ASD *deſcribed by the latter.*

Let the Tangent AB, which makes Right-Angles with BC, be put $= a$, BR$=x$, RS$=y$, and AS$=z$:



Then the Subtangent RT being $= \frac{y\dot{x}}{-\dot{y}}$, we have BT $= x + \frac{y\dot{x}}{-\dot{y}}$: Moreover, ſince the Diſtances BT and AS gone over in the ſame time, are as the Celerities n and 1, we alſo have BT $(= n \times \text{AS}) = nz = x + \frac{y\dot{x}}{-\dot{y}}$: Whence, in Fluxions (making $\dot{y}$ conſtant) $\frac{-y\ddot{x}}{\dot{y}} = n\dot{z}$; and conſequently $\frac{-n\dot{y}}{y} \left(= \frac{\ddot{x}}{\dot{z}}\right) = \frac{\ddot{x}}{\sqrt{\dot{y}^2 + \dot{x}^2}}$:

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The Fluent of which (*by Art.* 126.) is $-n \times$ Log. y $=$ Log. $\frac{\dot{x}+\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{y}}$: But when $y=a$, $\dot{x}$ is $=0$, and then the Equation becomes $-n \times$ Log. $a=0$; therefore the Fluent, duly corrected, is $n \times$Log. $a-n$ $\times$ Log. $y=$ Log. $\frac{\dot{x}+\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{y}}$, or Log. $\frac{a^n}{y^n}=$ Log. $\frac{\dot{x}+\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{y}}$. Whence it is evident that $\frac{a^n}{y^n}$ $=\frac{\dot{x}+\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{y}}$, and $\frac{a^n\dot{y}}{y^n}-\dot{x}=\sqrt{\dot{y}^2+\dot{x}^2}$; from which, by ſquaring both Sides, $2\dot{x}$ is found $=$ $\frac{a^n\dot{y}}{y^n}-\frac{y^n\dot{y}}{a^n}$; whoſe Fluent is $2x=-\frac{a^n y^{1-n}}{1-n}+$ $\frac{a^{-n}y^{n+1}}{n+1}$. But when $y=a$, x is $=0$, and then, $0=-\frac{a}{1-n}+\frac{a}{n+1}=-\frac{2na}{1-nn}$; therefore the Fluent corrected is $2x=-\frac{a^n y^{1-n}}{1-n}+\frac{a^{-n}y^{n+1}}{n+1}+$ $\frac{2na}{1-n^2}$. *Q. E. I.*

Otherwiſe (without ſecond Fluxions.)

441. Put ST $=P$ and RT $=Q$. Then ſince the abſolute Velocity of the Body *S* is denoted by Unity, that with which the Ordinate SR is carry'd towards the Body *T* will be denoted by $\frac{Q}{P}\times 1$ or $\frac{Q}{P}$ (*by Art.* 35.) which ſubtracted from n the Velocity of *T*, leaves $n-$ $\frac{Q}{P}$ for the relative Celerity with which *T* recedes from

R: After the ſame Manner, if from $\frac{Q}{P}\times n$ the Celerity of *T* in the Direction ST produced, there be taken (1) the Celerity of *S* in the ſame Direction, the Remainder, $\frac{nQ}{P} - 1$, will be the Celerity with which *T* recedes from *S*: Therefore, the Fluxions of Quantities being as the Celerities of their Increaſe, we have $n - \frac{Q}{P} : \frac{nQ}{P} - 1 :: \dot{Q} : \dot{P}$; and conſequently $\overline{nQ - P}\times\dot{Q} = \overline{nP - Q}\times\dot{P}$. But, ſince the Quantities P and Q are concerned exactly alike, the Equation thus derived will, in all probability, become more ſimple, by ſubſtituting for their Sum and Difference: Let therefore $P + Q = s$, and $P - Q = v$, or, which is the ſame, let $P = \frac{s+v}{2}$, and $Q = \frac{s-v}{2}$: Then, by Subſtitution, we ſhall have $\frac{ns - nv - s - v}{2} \times \frac{\dot{s} - \dot{v}}{2} = \frac{ns + nv - s + v}{2} \times \frac{\dot{s} + \dot{v}}{2}$; which contracted, &c. becomes $\overline{1+n}\times v\dot{s} = \overline{1-n}\times s\dot{v}$, or $\overline{1+n}\times \frac{\dot{s}}{s} = \overline{1-n} \times \frac{\dot{v}}{v}$; whoſe Fluent (corrected) is $\overline{1+n} \times \text{Log.}\ s = \overline{1-n}\times\text{Log.}\ v + 2n\times\text{Log.}\ a$, or $\text{Log.}\ s^{1+n} = \text{Log.}\ a^{2n}v^{1-n}$. Whence $s^{1+n} = a^{2n}v^{1-n}$, and conſequently $s^{1+n}\times v^{1+n} = a^{2n}v^2$: But $sv\ (= \overline{ST + RT} \times \overline{ST - RT} = RS^2) = y^2$; therefore $s^{1+n} \times v^{1+n} = y^{2n+2} = a^{2n}v^2$, and $v = \frac{y^{n+1}}{a^n}$; whence $s \left(= \frac{y^2}{v}\right) = \frac{a^n}{y^{n-1}}$, ST $\left(\frac{s+v}{2}\right) \frac{a^n}{2y^{n-1}} + \frac{y^{n+1}}{2a^n}$, RT $\left(\frac{s-v}{2}\right)$

$= \frac{a^n}{2y^{n-1}} - \frac{y^{n+1}}{2a^n}$. But RS (y) : RT $\left(\frac{a^n}{2y^{n-1}} - \frac{y^{n+1}}{2a^n}\right)$ $:: \dot{y} : \dot{x}$; whence $2\dot{x} = \frac{a^n\dot{y}}{y^n} - \frac{y^n\dot{y}}{a^n}$, and $2x = -\frac{a^n y^{1-n}}{1-n} + \frac{a^{-n} y^{n+1}}{n+1} + \frac{2na}{1-nn}$, the very ſame as before.

COROLLARY.

442. If the Velocity of *S* be greater than that of *T* (or n be leſs than Unity) the two Bodies will concur when the latter has moved over a Diſtance expreſſed by $\frac{na}{1-n^2}$; becauſe, when y becomes $= 0$, $2x$ is barely $= \frac{2na}{1-n^2}$. But if the Velocity of *S* be leſs than that of *T*, it is plain that *S* can never come up with *T*: But its neareſt Approach will be when $y = \overline{\frac{n-1}{n+1}}\Big|^{\frac{1}{2n}} \times a$: For, ſince ST is univerſally $= \frac{a^n}{2y^{n-1}} + \frac{y^{n+1}}{2a^n}$, let the Fluxion of this Expreſſion be taken and put equal to Nothing; and y will be found as above exhibited.

If the Celerities of *S* and *T*, inſtead of being uniform, vary according to a given Law; then, denoting the former by *A* and the latter by *B*, the Equation of the Curve will be $\frac{\ddot{x}}{\sqrt{\dot{y}^2 + \dot{x}^2}} = -\frac{B\dot{y}}{Ay}$: And if the Fluent of $-\frac{B\dot{y}}{Ay}$ be explicable by a Logarithm, as L. N; then, the Fluent of $\frac{\ddot{x}}{\sqrt{\dot{y}^2+\dot{x}^2}}$ being L. $\frac{\dot{y}+\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{y}}$ *, we ſhall have $N=\frac{\dot{y}+\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{y}}$; which, order'd, gives $\dot{x}=\frac{N\dot{y}}{2}-\frac{\dot{y}}{2N}$: Whence x will be found.

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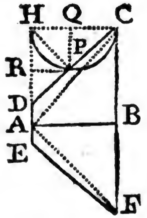
* Art. 126.

PROB. XVI.

443. *To determine the Fruſtum* CDEF *of a Triangular-Priſm, of a given Baſe* CF *and Altitude* BA; *which, moving in a Medium, in the Direction of its Length* BA, *ſhall be reſiſted the leaſt poſſible.*



Draw CH parallel to BA meeting ED, produced, in H: Moreover, let HP, PQ and PR be perpendicular to CD, CH and DH reſpectively.

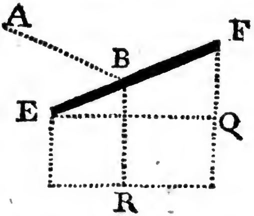
Since the Number of reſiſting Particles acting upon DC is as DH, and the Force of each as $\left(\frac{DR^2}{DP^2}\right)$ the Square of the Sine of the Angle of Incidence DPR, the whole Reſiſtance ſuſtained by DC will therefore be expreſſed by $\frac{DH\times DR^2}{DP^2}$, or DR, which is equal to it (by the Similarity of the Triangles DHP and DPR) Whence the Reſiſtance upon ADC is truly expreſſed by AR (AD + DR) and is a *Minimum* when its Defect (PQ) below the given Quantity AH (or BC) is a *Maximum*: But PQ is a *Maximum* when CQ and HQ are equal; becauſe, the Angle CPH being Right, a Semi-circle deſcribed upon CH will always paſs thro' the Point P; and it is well known that the greateſt Ordinate in a Semi-circle is That which divides the Diameter into two equal Parts.

Hence the Angle DCH, when the Reſiſtance upon ADC is a *Minimum*, will be juſt the Half of a Right-Angle, provided BC be given greater than BA; otherwiſe,

wiſe, the whole Priſm CAF will be leſs reſiſted than any Fruſtum CDEF of a greater Priſm.

PROB. XVII.

444. *To determine the Angle* RBE *which a Plane* EBF *muſt make with the Wind blowing in a given Direction* RB, *ſo that the Plane itſelf may be urged in another given Direction* BA *with the greateſt Force poſſible.*



It is known, from the Reſolution of Forces, that the Force whereby the Plane EF is urged in the given Direction BA, by a Particle of Air, acting in the Direction RB, is directly as the Rectangle of the Sines of the Angles (ABE, RBE) which the two given Directions make with the Plane: Therefore, ſince the Number of Particles acting on EF is as the Sine of RBE, it follows that the whole Force, or Effect, of the Wind, in the Direction BA, will be as $S.\mathrm{ABE}\times Squ.\,S.\,\mathrm{RBE}$; which being a *Maximum*, we have *(by Prob.* 5.*)* 3 : 1 :: Sine of the whole given Angle RBA : Sine of $\overline{\mathrm{RBE}-\mathrm{ABE}}$. Whence the Angles RBE and ABE are both given. *Q. E. I.*

Corollary.

445. If the Angle RBA be a Right one (which is the Caſe with regard to the Sails of a Windmill) then the Sine of $\overline{\mathrm{RBE}-\mathrm{AEB}}$ being $=\frac{1}{3}=,333$ &c. we ſhall have $\mathrm{RBE}-\mathrm{AEB}=19^\circ:28'$; and conſequently

$$\mathrm{RBE}\ \left(\frac{\mathrm{RBA}+\mathrm{AEB}}{2}\right)=54^\circ:44'.$$

PROB. XVIII.

446. *If two Bodies* A *and* B, *joined by a String, be urged in oppoſite Directions, towards* P *and* Q, *by any given Forces* F *and* f, *uniformly continu'd; 'tis propoſed to find the Tenſion of the String, or the Force whereby the Bodies endeavour to recede from each other.*

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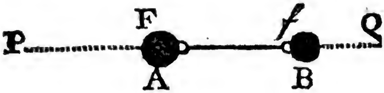
Since $F-f$ is the abſolute Force by which the two Bodies are, conſtantly, urged towards P, the *whole* Motion, generated in Both, in any Time T, will therefore be expreſſed by $\overline{F-f}\times T$: Whence, becauſe both Bodies (by reaſon of the String) acquire the ſame Velocity, the Motion generated in A, alone, will be $\frac{A}{A+B}\times\overline{F-f}\times T$, or that Part of the *Whole* defined by $\frac{A}{A+B}$. But the Motion of A, had it not been retarded by the String (or B) would have been $F\times T$; therefore the Loſs of Motion, by the Action



upon the String, is $F\times T-\frac{A}{A+B}\times\overline{F-f}\times T$, $=\frac{fA+FB}{A+B}\times T$: Which, divided by the Time T (wherein that Loſs or Effect is produced) gives $\frac{fA+FB}{A+B}$, for the Tenſion of the Thread, or the Force ſufficient to cauſe the ſaid Loſs of Motion.

The ſame otherwiſe.

447. Becauſe the Force F, was it to act alone, would communicate, by means of the String, the ſame Velocity to B as to A, the Part therefore of the Force F employ'd upon B, by which the String is ſtrech'd, will be $\frac{B}{A+B}\times F$, or $\frac{BF}{A+B}$: And, from the very ſame Argument, if the Force f was to act alone, the Tenſion of the Thread would be $\frac{fA}{A+B}$: Therefore, when both

the Forces act together, the Tenſion will be $\frac{fA+BF}{A+B}$: For it is very plain that, their acting both at the ſame time, no way influences their reſpective Effects on the Thread.

Q. E. I.

COROLLARY.

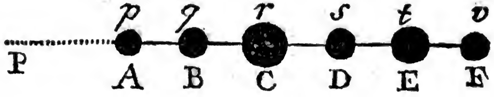
448. If the Forces F and f be reſpectively expounded by the Maſſes, or Weights, of the Bodies A and B; the Tenſion of the Thread will then become $\frac{2AB}{A+B}$. Whence it appears that the Tenſion of a Thread ſliding over a Pin or Pulley, by means of two unequal Weights A and B, ſuſpended at the Ends thereof, is equal to $\frac{2AB}{A+B}$: The Double whereof, or $\frac{4AB}{A+B}$, is the Weight which the Pin or Pulley ſuſtains, while the Bodies are in Motion; becauſe the Thread hangs double, or on both Sides the Pulley.

If ſeveral Bodies A, B, C, D *&c.* communicating by means of a String or Wire AF, be urged towards a Point P, in the Direction of the String or Wire, by any given Forces p, q, r, s *&c.* reſpectively, the Tenſion of the Part AB will be

$$=\frac{p\times\overline{B+C+D\ \&c.}-A\times\overline{q+r+s\ \&c.}}{A+B+C+D\ \&c.};$$

of the Part BC

$$=\frac{\overline{p+q}\times\overline{C+D+E\ \&c.}-\overline{A+B}\times\overline{r+s+t\ \&c.}}{A+B+C+D\ \&c.};$$



of CD

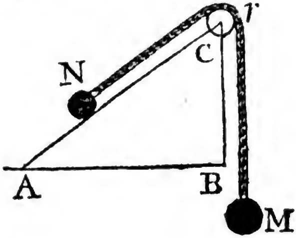
$$=\frac{\overline{p+q+r}\times\overline{D+E+F}-\overline{A+B+C}\times\overline{s+t+v}}{A+B+C+D\ \&c.};$$

&c. &c.

All which eaſily follows from above; and will anſwer alſo in thoſe Caſes where ſome of the Forces are ſuppoſed to act in the contrary Direction, if every ſuch Force be conſidered as a negative Quantity.

PROB. XIX.

449. *Let it be required to raiſe a given Weight* N, *to a given Height* BC, *along an inclin'd Plane* AC, *by means of another given Weight* M, *connected to the former by a flexible Rope* N*r*M, *moving over a Pulley at* C; *to find the Tenſion of the Rope; alſo the Inclination and Length of the Plane, ſo that the Time of the whole Aſcent may be the leaſt poſſible.*



It is well known that the Force by which N tends to deſcend along the Plane AC, or acts in oppoſition to M (ſuppoſing BC$=a$, and AC $=x$) will be $\frac{aN}{x}$: Therefore $M-\frac{aN}{x}$, or $\frac{xM-aN}{x}$ is the efficacious Force, by which the Bodies are accelerated: But it is likewiſe demonſtrable that the Time of deſcribing any Line by means of a Velocity uniformly accelerated, is in the ſubduplicate Ratio of the Length thereof, directly, and the ſubduplicate Ratio of the acceleratihg Force, inverſely *: Whence it follows that the Time of deſcribing AC will be repreſented by $\frac{x}{\sqrt{xM-aN}}$: Whoſe Fluxion (or that of its Square) being made equal to Nothing, x will be found $=\frac{2aN}{M}$, or $M:2N::a:x$. Hence the

Time

* Art. 203.

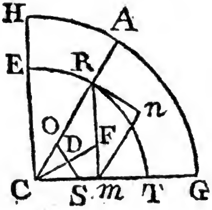
Time of the Aſcent will be the leaſt poſſible, when the Sine of the Plane's Inclination is to the Radius, as the Power (M) is to twice the Weight (N) to be raiſed.

The Tenſion of the Rope will be determined from the laſt Problem, (by writing N for A, $\frac{aN}{x}$ for F, M for B, and M for f) and comes out $= \frac{MN}{M+N} \times \frac{a+x}{x}$.

Q. E. I.

PROB. XX.

450. *Let* AC *repreſent a Piece of Timber, moveable about a Center* C, *making any Angle* ACG *with the Plane of the Horizon* CG; *to determine the Poſition of a Prop or Supporter* OS, *of a given Length, which ſhall ſuſtain it with the greateſt Facility, in any given Poſition; and alſo what Inclination* AC *will have to the Horizon when the leaſt Force that can ſuſtain it, is greater than the leaſt Force in any other Poſition.*



Let R be the Center of Gravity of the Beam AC, and let Rn, Rm and CD be perpendicular to AC, CG and OS reſpectively: Putting SO$=a$, CR $=r$, C$m=x$, and the Weight of the Beam $=w$.

Then, *by the Principles of Mechanics*, we ſhall have, firſt, as Rm : Rn, or as, $r : x :: w : \left(\frac{xw}{r}\right)$ the Force, which acting at R, in the Direction Rn, is ſufficient to ſuſtain the Beam AC; ſecondly, as CO : CR (r) :: $\frac{xw}{r}$ (the Quantity laſt found) : $\frac{xw}{\text{CO}}$, the Force able to ſupport it, at O, in a perpendicular Direction; and, laſtly, as CD : CO :: $\frac{xw}{CO} : \frac{xw}{CD}$, the Force, or Weight, actually ſuſtained by the given Prop SO. Which Force will therefore be the leaſt poſſible when the Perpendicular CD is the greateſt poſſible, let the Angle of Inclination GCA be what it will: But of all Triangles, having the ſame Baſe (OS) and vertical Angle (SCO) the Iſoſceles one is known to have the greateſt Perpendicular: Therefore the Triangle CSO will be Iſoſceles, and the Angles S and O equal to each other, when the Weight ſuſtain'd by the Prop OS is a *Minimum*.

But, now, to give a Solution to the latter Part of the Problem, or to find (ſuppoſing the Angles S and O to be equal) when $\frac{x}{CD} \times w$ is a *Maximum*, let CD produced meet *m*R in F; and then, becauſe of the ſimilar Triangles CDS and C*m*F, we ſhall have CD : *x* (C*m*) :: SD ($\frac{1}{2}a$) : *m*F, or $\frac{x}{CD} = \frac{mF}{\frac{1}{2}a}$; and conſequently $\frac{x}{CD} \times w = \frac{mF}{\frac{1}{2}a} \times w$: But, ſince CF biſects the Angle *m*CR, we alſo have, $r+x$ (CR+C*m*) : *x* (C*m*) :: $\sqrt{r^2-x^2}$ (R*m*) : F*m* $= \frac{x\sqrt{r^2-x^2}}{r+x} = x\sqrt{\frac{r-x}{r+x}}$: Whence the Force $\frac{mF}{\frac{1}{2}a} \times w$, acting upon the Supporter, is likewiſe truly expreſſed by $\frac{wx}{\frac{1}{2}a}\sqrt{\frac{r-x}{r+x}}$: Whereof the Fluxion being taken and put equal to Nothing &c. we get $x = \frac{r\sqrt{5}-r}{2}$: Therefore CR : C*m* (:: 1 : $\frac{\sqrt{5}-1}{2}$) :: Radius : Coſine of RCG=51° : 50′, the Inclination required.

PROB.

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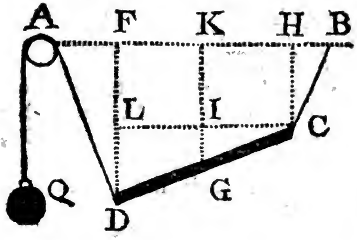
PROB. XXI.

451. *To determine the Poſition of a Beam* CD, *moveable about one End* C *as a Center, and ſuſtained at the other End* D *by a given Weight* Q, *appended to a Cord* QAD *paſſing over a Pulley at a given Point* A.



Let G be the Center of Gravity of the Beam; alſo let DF, GK and CH be perpendicular to the Plane of the Horizon, and CL and AH parallel to the ſame: Putting AH$=a$, CH$=b$, CD$=c$, CG$=d$, DL$=x$, CL $=y$, and the Weight of the Beam $=w$. Then AF $=a-y$, DF $=b+x$, and AD $(\sqrt{AF^2+DF^2})=\sqrt{a^2-2ay+y^2+b^2+2bx+x^2}$; which (becauſe $y^2+x^2=c^2$) will alſo be $=\sqrt{a^2+b^2+c^2+2bx-2ay}=\sqrt{f^2+2bx-2ay}$ (by putting $f^2=a^2+b^2+c^2$) whoſe Fluxion, $\frac{b\dot{x}-a\dot{y}}{\sqrt{f^2+2bx-2ay}}$, multiply'd by Q, is the Momentum of the Weight Q, ſuppoſing the Beam to to be in Motion. Moreover, becauſe DC : DL :: CG : GI, we have GI $=\frac{dx}{c}$; whoſe Fluxion, $\frac{d\dot{x}}{c}$, multiply'd by w, is the Momentum of the Beam itſelf in a vertical Direction.

Wherefore making theſe *Momenta* equal to each other (according to the Principles of Mechanics) we get $\frac{b\dot{x}-a\dot{y}}{\sqrt{f^2+2bx-2ay}}\times Q=\frac{d\dot{x}}{c}\times w$, and conſequently $\overline{b\dot{x}-a\dot{y}}\times cQ=dw\dot{x}\sqrt{f^2+2bx-2ay}$: But, ſince y^2+

$y^2 + x^2 = c^2$, we have $2y\dot{y} + 2x\dot{x} = 0$, or $-\dot{y} = \frac{x\dot{x}}{y}$: And therefore (by Subſtitution) $\overline{b\dot{x} + \frac{ax\dot{x}}{y}} \times c\mathcal{Q} = dw\dot{x}\sqrt{f^2 + 2bx - 2ay}$, or $\overline{by + ax} \times c\mathcal{Q} = dyw \times \sqrt{f^2 + 2bx - 2ay}$: From whence, and the foregoing Equation $x^2 + y^2 = c^2$, both x and y may be determined.

The ſame otherwiſe.

452. It is evident, from Mechanics, that the Force which, acting in the Direction DF, would ſuſtain the End D, is to the whole Weight w, as CG to CD; and therefore is $= \frac{\text{CD}}{\text{CG}} \times w$: It is likewiſe known that two Forces acting in the different Directions DF and DA, ſo as to have the ſame Effect in ſuſtaining DC, or cauſing It to move about the Point C, muſt be to each other, inverſely, as the Sines of the Angles of Incidence FDC and ADC. Therefore we have $S.\,\text{FDC} : S.\,\text{ADC} :: \mathcal{Q} : \frac{\text{CD}}{\text{CG}} \times w$; from which given Ratio of the Sines, the Angles themſelves will be found, by an algebraic Proceſs independent of Fluxions.

COROLLARY.

453. If the Poſition of CD be ſuppoſed given, and the Tenſion of AD (or the Weight $\mathcal{Q}$) be required: Then, from the foregoing Proportion, we ſhall have $\mathcal{Q} = \frac{S.\,\text{FDC}}{S.\,\text{ADC}} \times \frac{\text{CG}}{\text{CD}} \times w$. Which will alſo expreſs the Tenſion of AD when the End C is ſuſtained by a Cord BC inſtead of a Pin at C: Whence it follows that the Tenſions of two Cords AD and BC, ſuſtaining a Beam or Rod CD, at its Extremes D and C, are expreſſed by $\frac{S.\,\text{FDC}}{S.\,\text{ADC}} \times \frac{\text{CG}}{\text{CD}} \times w$, and $\frac{S.\,\text{HCD}}{S.\,\text{BCD}} \times \frac{\text{DG}}{\text{CD}} \times w$; and there-

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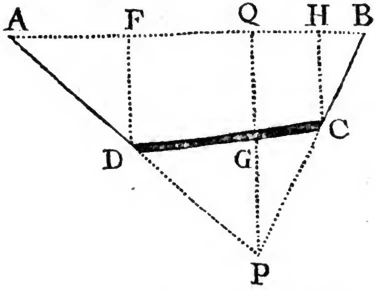
therefore are to each other as $\frac{CG}{S.\ ADC}$ to $\frac{DG}{S.\ BCD}$, or as $S.\ BCD \times CG$ to $S.\ ADC \times CG$ reſpectively; becauſe the Sine of FDC and that of its Supplement HCD are equal to each other.

PROB. XXII.

454. *To determine the Poſition of a Beam* DC, *ſuſpended at its Extremes by two Cords* AD *and* BC *of given Lengths, from two given Points* A *and* B *in the ſame horizontal Line* AB.

Let G be the Center of Gravity of the Beam, and let DF and CH be perpendicular to AB.



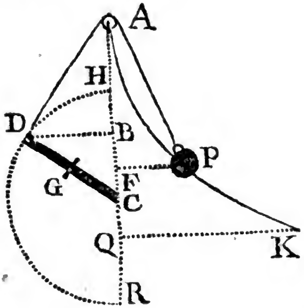
It appears, from the Corol. to the laſt Problem, that the Tenſion of AD is to that of BC, as $\frac{CG}{S.\ ADC}$ to $\frac{DG}{S.\ BCD}$; whence (by the Reſolution of Forces) the Force of AD, in a Direction parallel to the Horizon, is to the Force of BC, in the oppoſite Direction, as $\frac{CG}{S.\ ADC} \times \frac{S.\ ADF}{Rad.}$ to $\frac{DG}{S.\ BCD} \times \frac{S.\ BCH}{Rad.}$. Which Forces, that the Beam may remain *in Equilibrio*, muſt conſequently be equal to each other; and therefore $\frac{S.\,BCD}{S.\,ADC}=\frac{S.\,BCH}{S.\,ADF}\times\frac{DG}{CG}$. But now, to determine the Angles themſelves, from this Equation and the given Lengths of AB, BC *&c.* let AD and BC be produced to meet each other in P, and let PQ, perpendicular to AB, be drawn; putting $AB=a$, $AD=b$, $BC=c$, $DC=d$, $DG=f$, $CG=g$, $AP=x$ and $BP=y$.

Then, becauſe $AB : AP+BP :: AP-BP : AQ-BQ$ $=\frac{AP^2-BP^2}{AB}$, we have $AQ=\frac{1}{2}AB+\frac{AP^2-BP^2}{2AB}$ $=\frac{AB^2+AP^2-BP^2}{2AB}$; and conſequently the Co-ſine of A ($=$ Sine ADF) to the Radius $1=\frac{AB^2+AP^2-BP^2}{2AB\times AP}$: Whence, from the ſame Argument, it is evident that the Co-ſine of B ($=$ Sine BCH) will be expreſſed by $\frac{AB^2+BP^2-AP^2}{2AB\times BP}$; and That of APB by $\frac{AP^2+BP^2-AB^2}{2AP\times BP}$; and alſo by $\frac{PD^2+PC^2-DC^2}{2PD\times PC}$; which two laſt Quantities being equal to each other, we have $PD\times PC\times\overline{AP^2+BP^2-AB^2}=AP\times BP\times\overline{PD^2+PC^2-DC^2}$; that is $\overline{x-b}\times\overline{y-c}\times\overline{x^2+y^2-a^2}=xy\times\overline{\overline{x-b}^2+\overline{y-c}^2-d^2}$. Moreover, ſince $PC : PD :: S.\,ADC$ (or PDC) $: S.\,BCD$ (or PCD) we alſo have $\frac{PD}{PC}=\frac{S.\,BCD}{S.\,ADC}=\frac{S.\,BCH}{S.\,ADF}\times\frac{DG}{CG}$ (by the firſt Equation); whence $CG\times PD\times S.\,ADF=DG\times PC\times S.\,BCH$; that is $CG\times PD\times\frac{AB^2+AP^2-BP^2}{2AB\times AP}=DG\times PC\times\frac{AB^2+BP^2-AP^2}{2AB\times BP}$, or

$CG\times PD\times BP\times\overline{AB^2+AP^2-BP^2}=DG\times PC\times AP\times\overline{AB^2+BP^2-AP^2}$, which, in algebraic Terms, is $gy\times\overline{x-b}\times\overline{a^2+x^2-y^2}=fx\times\overline{y-c}\times\overline{a^2+y^2-x^2}$. From whence and the preceding Equation the Values of x and y will be known.

PROB. XXIII.

455. *Suppoſing a Beam* CD, *moveable about one End* C, *as a Center, to be ſuſtained at the other End* D *by means of a given Weight* P, *hanging at a Rope paſſing over a Pulley at a given Point* A, *vertical to* C; *'tis propoſed to find the Curve* APK *along which the Weight muſt aſcend, or deſcend, ſo as to be, every where, a juſt Counterpoiſe to the Beam.*



From the Center C, with the Radius CD, let a Semi-circle HDR be deſcribed, and let DB and PF be perpendicular to the vertical Line AHCR; alſo let CD$=a$, CA$=b$, AH$=c$, AF$=x$, PF$=y$, HB$=z$, and the Length of the Rope DAP$=m$; likewiſe let HQ (h) be the given Value of x (AF) when D coincides with H.

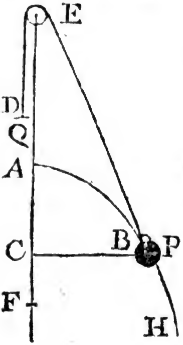
Becauſe the Weight and the Beam are always in *Equilibrio*, by Hypotheſis, their *Momenta*, and conſequently their Velocities, in a vertical Direction, muſt be every where in a conſtant Ratio; and therefore the Diſtance QF $(h-x)$ aſcended by the Weight *P*, will be, to the Diſtance HB deſcended by the End of the Beam D likewiſe in a conſtant Ratio: Let this Ratio be that of b to any given Quantity d, that is, let $h-x:z::b:d$, and we ſhall have $dh-dx=bz$: Moreover, we have AD^2 $(CD^2+AC^2-2AC\times BC)=a^2+b^2-2b\times\overline{a-z}=\overline{b-a}^2+2bz=c^2+2bz=c^2-2dh+2dx$: Whence AP $(m-AD)=m-$

$\sqrt{cc-2db+2dx}$, and therefore, y^2 $(AP^2-AF^2) = \overline{m-\sqrt{cc-2db+2dx}}|^2 - x^2$. *Q. E. I.*

After the ſame manner a Curve may be found, along which a Weight deſcending, ſhall be every where in *Equilibrio* with another Weight aſcending thro' the Arch of a given Curve.

PROB. XXIV.

456. *To find the Equation of a Curve* ABH, *along which a given Weight* P, *ſuſpended by a String* PED *paſſing over a Pulley* E, *muſt deſcend, ſo that the Tenſion of the String may vary according to any given Law.*



Let EC be perpendicular, and CP parallel, to the Plane of the Horizon; alſo let $AE=a$, $AC=x$, $CB=y$, $EP=v$, and let the Tenſion of the String (or the Force acting at the End D) be denoted by any variable, or conſtant, Quantity Q.

Therefore, becauſe the Celerity of the Weight P, in a vertical Direction, is to its Celerity, in the Direction EP produced, (or the Celerity of the other End D) as $\dot{x}$ to $\dot{v}$, it is evident that the Weight itſelf muſt be to the tending Force Q, inverſely in that Ratio, and conſequently $P\dot{x}=Q\dot{v}$.

Furthermore, becauſe $EC=a+x$ and $BC^2=BE^2-EC^2$, we have $y^2=v^2-\overline{a+x}|^2$: From which Equations, when the Relation of P and Q is given, the Curve itſelf will alſo be known.

Thus, for Example, let the Ratio of P to Q, be conſtant, or that of m to n, then $m\dot{x}$ being $=n\dot{v}$, we have (by taking the Fluent) $mx+na=nv$; whence $v=a+\frac{mx}{n}$; and therefore y^2 $(=a^2+\frac{2max}{n}+\frac{m^2x^2}{n^2}$

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$-a^2-2ax-x^2)=\frac{m-n}{n}\times 2ax+\frac{m^2-n^2}{n^2}\times x^2$: Which is the Equation of an Hyperbola.

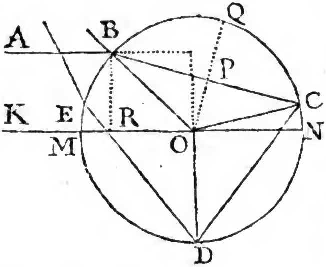
Again, for a ſecond Example, let the tending Force $\mathcal{Q}$ be to the Weight P, as DE^n to $\mathrm{AC}^m\times c^{n-m}$, or as $\overline{b-v}|^n : x^m c^{n-m}$ (ſuppoſing $b=$PED and $c=$ any given Line AF) Therefore, ſince $\mathcal{Q}=\frac{\overline{b-v}|^n}{c^{n-m}x^m}\times P$, and $\frac{\overline{b-v}|^n}{c^{n-m}x^m}\times P\dot{v}$ $(=\mathcal{Q}\dot{v})=P\dot{x}$, we have $\overline{b-v}^n\times\dot{v}$ $=c^{n-m}x^m\dot{x}$, and ſo $\frac{\overline{b-a}|^{n+1}-\overline{b-v}|^{n+1}}{n+1}=$ $\frac{c^{n-m}x^{m+1}}{m+1}$; whence $\overline{b-v}|^{n+1}=\overline{b-a}|^{n+1}-$ $\frac{\overline{n+1}\times c^{n-m}x^{m+1}}{m+1}$, and v (EP) $=b-$ $\overline{\overline{b-a}|^{n+1}-\frac{\overline{n+1}\times c^{n-m}x^{m+1}}{m+1}}|^{\frac{1}{n+1}}$. From which the Relation of x and y, or the Value of BC, is alſo known.

But if $m=0$, and $n=1$, (which will be the Caſe when the Force acting at D is equal to that by which a Beam or Rod is made to move about a Center, as in the laſt Problem) v will then become, barely, $=b-$ $\overline{\overline{b-a}|^2-2cx}|^{\frac{1}{2}}$, and therefore y^2 $(=v^2-\overline{a+x}|^2)$ $=\overline{b-\sqrt{\overline{b-a}|^2-2cx}}|^2-\overline{a+x}|^2$: Therefore ABH is, in this Caſe, a Line of the fourth Order.

PROB.

PROB. XXV.

457. *Suppoſing a Ray of Light* ABCD *to be refracted at the Surface of a given Sphere* MQND, *and afterwards reflected any given Number (n) of Times, within the Sphere; to determine the Diſtance of the Incident Ray* AB *from the Axis* MN, *ſo that the Arch* MBCDE, *intercepted by the given Point* M *and the emerging Ray at* E, *may be a* Minimum.



Let the Radius OB $= 1$, the Sine of Incidence BR $= x$, and the Sine of Refraction OP $= y$, and let the given Ratio of the two laſt be that of p to q.

Since all the Angles of Incidence and Reflexion BCO OCD, CDO &c. are equal, the Arcs BC, CD and DE muſt alſo be equal; and conſequently MBCDE $=$ MB $+ \overline{n+1} \times$ BC $=$ MB $+ \overline{2n+2} \times$ BQ:
*Art. 22. Whoſe Fluxion is to be equal to Nothing *. Now the Fluxion of the Arch MB, whoſe Sine is x and
† Art. 142. Radius Unity, will be $= \frac{\dot{x}}{\sqrt{1-x^2}}$ †; and that of the Arch BQ, whoſe Co-ſine (OP) is y, $= \frac{-\dot{y}}{\sqrt{1-y^2}}$. Hence we have $\frac{\dot{x}}{\sqrt{1-x^2}} - \frac{\overline{2n+2} \times \dot{y}}{\sqrt{1-y^2}} = 0$: But, ſince $x : y :: p : q$, y is $= \frac{qx}{p}$ and $\dot{y} = \frac{q\dot{x}}{p}$; and ſo we have $\frac{\dot{x}}{\sqrt{1-x^2}} - \frac{\overline{2n+2} \times q\dot{x}}{\sqrt{p^2-q^2x^2}} = 0$; whence (putting

$m =$

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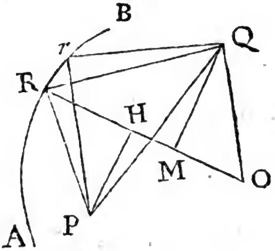
$m=2n+2)$ x is found $=\frac{1}{q}\sqrt{\frac{m^2q^2-p^2}{m^2-1}}$: From which it is obſervable, that, when mq is leſs than p, or $2n+2$ leſs than $\frac{p}{q}$, the Arch MBCD continually increaſes with BM; and therefore is the leaſt poſſible, when B coincides with M. *Q. E. I.*

PROB. XXVI.

458. *If two Rays of Light* PR *and* P*r*, *from a given Point* P, *making an indefinitely ſmall Angle with each other, be reflected at a given Curve Surface* ARB; *'tis propoſed to determine the Concourſe, or Focus,* Q *of the reflected Rays* RQ *and r*Q.



Let RO, perpendicular to the Curve, be the Radius of a Circle having the ſame Curvature with ARB at R; make PH and QM perpendicular to RO, join Q, O; and put RO $=r$, PR$=y$, RH$=v$, and RQ$=z$.

Then, becauſe the Angle of Reflection ORQ is equal to the Angle of Incidence ORP, the Triangles RQM and RPH will be ſimilar, and therefore $y:v::z:$ RM $=\frac{vz}{y}$: Whence OQ^2 (RO^2 + RQ^2 − 2RO × RM) $=r^2+z^2-\frac{2rvz}{y}$.

But, ſince this Quantity OQ^2 continues the ſame (by Hypotheſis) whether we regard one Ray or the other (that is, whether y ſtands for PR or Pr) its Fluxion muſt therefore be equal to Nothing; that

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is $2z\dot{z} - \frac{2r\dot{v}zy + 2rv\dot{z}y - 2rvz\dot{y}}{y^2} = 0$: Whence

$z = \frac{vy\dot{z}}{\frac{y^2\dot{z}}{r} + v\dot{y} - y\dot{v}}$: But (*by Art.* 35.) $\dot{z} = -\dot{y}$; therefore $z = \frac{-vy\dot{y}}{-\frac{y^2\dot{y}}{r} + v\dot{y} - y\dot{v}}$: Moreover (*by Art.* 73.) $r = \frac{y\dot{y}}{\dot{v}}$; therefore

$$z = \frac{-vy\dot{y}}{-y\dot{v} + v\dot{y} - y\dot{v}} = \frac{vy\dot{y}}{2y\dot{v} - v\dot{y}}.$$ *Q. E. I.*

Example 1. Let ARB be an Arch of the Logarithmic Spiral: Whoſe Equation is $av = by$ †: And then, $\dot{v}$ being $= \frac{b\dot{y}}{a}$, we ſhall have $z\ \left(\frac{vy\dot{y}}{2y\dot{v} - v\dot{y}}\right) = y$: Therefore in this Caſe the Incident and Reflected Rays are equal to each other.

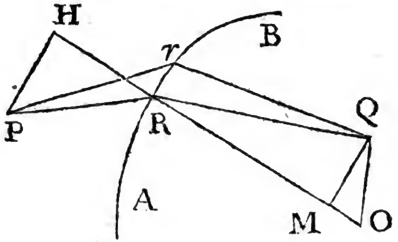
Ex. 2. Let ARB be ſuppoſed to degenerate into a Right-line: In which Caſe v being conſtant, its Fluxion $\dot{v}$ is $= 0$; and therefore $z\ \left(= \frac{vy\dot{y}}{-v\dot{y}}\right) = -y$: Which being negative, indicates that the Rays do not converge after Reflection, but, on the contrary, diverge from a Point on the contrary Side of ARB, at the Diſtance y. Which is very eaſy to demonſtrate by common Geometry.

PROB. XXVII.

459. *Let two Rays of Light* PR *and* P*r*, *from a given Point* P, *be refracted at a given Curve Surface* ARB; *to determine the Focus* Q *of the refracted Rays* RQ *and r*Q.

Let the Lines RO, RH *&c.* be drawn, and denoted as in the preceding Problem: Moreover, let the Sine of Incidence PRH (to the Radius 1) be repreſented by s, and let it be to the Sine of Refraction ORQ, in the given Ratio of 1 to n.

Then (*by Trigonometry*) 1 : ns (Sine QRM) :: z (RQ) : QM $= nsz$; and therefore RM $= \sqrt{z^2 - n^2s^2z^2} = z\sqrt{1-n^2s^2}$. From whence, following the Steps of the preceding Problem, we alſo get $OQ^2 = r^2 + z^2 - 2rz\sqrt{1-n^2s^2}$; and its Fluxion $2z\dot{z} - 2r\dot{z}\sqrt{1-n^2s^2} + \frac{2rzn^2s\dot{s}}{\sqrt{1-n^2s^2}} = 0$; or $z\dot{z}\sqrt{1-n^2s^2} - r\dot{z}\times\overline{1-n^2s^2} + n^2rzs\dot{s} = 0$. But (*by Art.* 35.) $\dot{z} = -n\dot{y}$; therefore $-z\dot{y}\sqrt{1-n^2s^2} + r\dot{y}\times\overline{1-n^2s^2} + nrzs\dot{s} = 0$: Moreover (*by Trig.*) 1 (Radius) : s (Sine of PRH) :: y (PR) : $\sqrt{y^2-v^2}$ (PH) whence we have $sy = \sqrt{y^2-v^2}$, $s^2 = 1 - \frac{v^2}{y^2}$, and $s\dot{s} = \frac{-y^2v\dot{v}+v^2y\dot{y}}{y^4} = \frac{v^2\dot{y} - yv\dot{v}}{y^3}$; which Values, of s^2 and $s\dot{s}$, being ſubſtituted in the foregoing Equation, it becomes $-z\dot{y}\sqrt{1-n^2+\frac{n^2v^2}{y^2}} + r\dot{y}\times\overline{1-n^2+\frac{n^2v^2}{y^2}} + nrz\times\frac{v^2\dot{y}-yv\dot{v}}{y^3} = 0$, or $-zy^2\dot{y}\sqrt{\overline{1-nn}\times y^2+n^2v^2} + ry\dot{y}\times\overline{\overline{1-nn}\times y^2+n^2v^2} + nrz\times\overline{v^2\dot{y}-yv\dot{v}} = 0$; or (putting $\sqrt{\overline{1-n^2}\times y^2+n^2v^2} = w$) $\frac{-zy^2w\dot{y}}{r} + w^2y\dot{y} +$

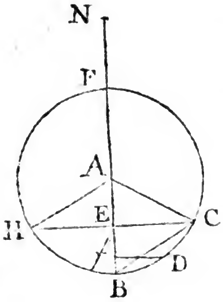


$nzv^2\dot{y} - nyzv\dot{v} = 0$. But (*by Art.* 73.) $r = \frac{y\dot{y}}{\dot{v}}$, therefore $-zyw\dot{v} + w^2y\dot{y} + nzv^2\dot{y} - nyzv\dot{v} = 0$, and conſequently $z = \frac{w^2y\dot{y}}{\overline{wy + nvy} \times \dot{v} - nv^2\dot{y}}$. *Q. E. I.*

From this Solution, that of the preceding Problem is eaſily derived: Alſo from hence the Cauſtic (or the Curve which is the Locus of all the Points Q thus found) will likewiſe be given.

PROB. XXVIII.

460. *To find the Time of the Vibration of a Pendulum in the Arch of a Circle.*



Let AB denote the Pendulum in a vertical Poſition; and from any Point D in the given Arch CBH, wherein the Vibrations are perform'd, draw Df parallel to CH; and let AB$=a$, BE$=c$, B$f$$=x$, and BD$=z$: By the Nature of the Circle we have $\dot{z} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$ *: Whence the Fluxion of the Time, being as $\frac{\dot{z}}{\sqrt{Ef}}$ †, will be defined by $\frac{a\dot{x}}{\sqrt{c - x} \times \sqrt{2ax - xx}}$

* Art. 142.

† Art. 207.

$$= \frac{a\dot{x}}{\sqrt{cx - xx} \times \sqrt{2a - x}} = \frac{\overline{\tfrac{1}{2}a}^{\frac{1}{2}} \times \dot{x}}{\sqrt{cx - xx}} \times \overline{1 - \frac{x}{2a}}^{-\frac{1}{2}}$$

$$= \frac{\overline{\tfrac{1}{2}a}^{\frac{1}{2}} \times \dot{x}}{\sqrt{cx - xx}} \times \overline{1 + \frac{x}{2\,.\,2a} + \frac{3x^2}{2\,.\,4\,.\,4a^2} + \frac{3\,.\,5x^3}{2.4.6.8a^3} + \frac{3\,.\,5\,.\,7x^4}{2\,.\,4\,.\,6\,.\,8\,.\,16a^4}} \text{ \&c.}$$

Whereof the Fluent,

when $x=c$, (or $\overline{cx-x^2}|^{\frac{1}{2}}=0$) is, (*by Art.* 142. *and* 286.) equal to

$$p\sqrt{\tfrac{1}{2}a}\times\overline{1+\frac{c}{2.2.2a}+\frac{3.3c^2}{2.2.4.4.\overline{2a}|^2}+\frac{3.3.5.5c^3}{2.2.4.4.6.6.\overline{2a}|^3}+\frac{3.3.5.5.7.7c^4}{2.2.4.4.6.6.8.8.\overline{2a}|^4}}$$

&c. Which therefore is proportional to the Time of half one Vibration; where p ſtands for the Semi-Periphery of the Circle whoſe Radius is Unity.

COROLLARY I.

461. Since the Time of the perpendicular Deſcent of a Body through any given Right-line u, computed according to the ſame Method, is as the Fluent of $\frac{\dot{u}}{\sqrt{u}}$ or $2\sqrt{u}$, it follows that the Time of falling along the Diameter BF ($2a$), or the Chord CB †, will † Art. 205. be truly defined by $2\sqrt{2a}$: Which therefore is to the Time of the Deſcent thro' the Arch CDB, as $\frac{4}{p}$ to $1+\frac{c}{2.2.2a}+\frac{3.3c^2}{2.2.4.4.\overline{2a}|^2}$, *&c.* From whence, as the Time of falling thro' the Diameter BF, is abſolutely given, *by Art.* 202. the true Time of Vibration will alſo be known.

COROLLARY II.

462. If the Arch in which the Pendulum vibrates be very ſmall, the above Proportion will become, nearly, as 4 to p: From which it appears, that the Time of Deſcent thro' any very ſmall Arch CB is to that along the Chord CB, as the Periphery of any Circle is to four times its Diameter.

COROLLARY III.

463. Hence, we have a Method for determining how far a Body freely deſcends in a given Time; by knowing

the Time of Vibration, of a given Pendulum: For, if BN be aſſumed for the Space thro' which a Body would deſcend during the Time of one whole Vibration, in the very ſmall Arch CBH; then, the Diſtances deſcended being as the Squares of the * Times we have, from the laſt Corollary, as $4^2 : \overline{2p}|^2 :: \text{BE}\ (2a) : \text{BN}$, or $1 : \frac{1}{2}p^2 : a : \text{BN}$; that is, as the Square of the Diameter of a Circle is to half the Square of its Periphery, ſo is the Length of the Pendulum, to the Diſtance a Body will freely deſcend, from Reſt, in the Time of one Oſcillation. Thus, for inſtance (becauſe it is found from Experiment that a Pendulum 39,2 Inches long vibrates Seconds) it will be as $1 : 4{,}934\ (=\frac{1}{2}p^2) :: 39{,}2 : 193$ Inches, the Diſtance which a heavy Body will fall in the firſt Second of Time.

* Art 201.

Corollary IV.

464. Moreover, from the foregoing Series, the Time which a Pendulum, vibrating in an exceeding ſmall Arch, will loſe when made to vibrate in a greater Arch of the ſame Circle may alſo be deduced:

For let T be put to denote the Number of Seconds in 24 Hours (or any other given Time) then the Number of Vibrations, performed in that Time will be as

$$\frac{T}{1+\frac{c}{2\,.\,2\,.\,2a}+\frac{3\cdot 3c^2}{2\,.\,2\,.\,4\,.\,\overline{2a}|^2\ \&c.}};$$

which, therefore, in an exceeding ſmall Arch (where c may be taken as Nothing) will be expreſſed by T: And ſo the Time (t) or Number of Vibrations loſt will be

$$T-\frac{T}{1+\frac{c}{2\,.\,2\,.\,2a}+\frac{3\cdot 3c^2}{2\,.\,2\,.\,4\,.\,4\,.\,\overline{2a}^2\ \&c.}}=T\times\overline{\frac{c}{8a}+\frac{5c^2}{256a^2}\ \&c.}$$

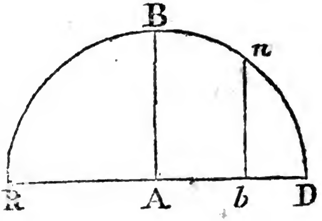
(by dividing by the Denominator.)

Now, if the Number of Degrees deſcribed on each Side of the Perpendicular be repreſented by D, the

Arch itſelf, on each Side, will be $= 3.14159$ &c. $\times a \times \frac{D}{180}$; which, if the Value of D be not more than about 15 or 20 Degrees, will be nearly equal to its Chord, repreſented by $\sqrt{2ac}$ $(=\sqrt{BF\times BE},)$ From which Equation we get $\frac{c}{a} = \frac{D^2}{6560}$: This Value, ſubſtituted above, gives $t = T \times \overline{\frac{D^2}{8\times 6560} + \frac{5D^4}{256\times\overline{6560}^2}}$ &c. $= T \times \frac{D^2}{52480}$ nearly: Which, when T is interpreted by 86400 Seconds (or one whole Day) becomes $= 1\frac{1}{2} \times D^2$, nearly: And ſo many are the Seconds which will be loſt *per Diem* in the Arch D. From whence we gather, that if the Pendulum meaſures true Time in any ſmall Arch, whoſe Degrees on each Side the Perpendicular are denoted by A, the Number of Seconds loſt *per Diem* in another Arch whoſe Degrees are B, will be nearly repreſented by $\frac{3}{2} \times \overline{B^2 - A^2}$: Thus, if a Pendulum meaſures true Time, in an Arch of 3 Degrees, it will loſe $10\frac{1}{2}$ Seconds a Day in an Arch of 4 Degrees, and 24″ in an Arch of 5 Degrees.

PROB. XXIX.

465. *To determine the* Meridional Parts *anſwering to any propoſed Latitude, according to* Wright's *Projection, apply'd to the true ſpheroidal Figure of the Earth.*



Let DAR be the Axis, AB the Semi-equatoreal Diameter, and DBR a Meridian, of the Earth; alſo let *bn* be an Ordinate to the Ellipſis DBR; putting AD(=AR)

$=1$, $BA=d$, $Ab=x$, $bn=y$, $Bn=z$, and the Meridional Diſtance (in Parts of the Semi-Axis AD) $=u$.

Then, by the Nature of the Ellipſis, we have $y=d\sqrt{1-x^2}$; therefore $\dot{y}=\frac{-dx\dot{x}}{\sqrt{1-x^2}}$; and conſequently $\dot{z}=\sqrt{\dot{x}^2+\frac{d^2x^2\dot{x}^2}{1-xx}}$: Which, by putting $b^2=d^2-1$, will be reduced to $\dot{z}=\frac{\dot{x}\sqrt{1+b^2x^2}}{\sqrt{1-x^2}}$. Whence, by the Nature of the Projection, it will be as bn $(d\sqrt{1-x^2})$: AB (d) :: $\dot{z}$ $\left(\frac{\dot{x}\sqrt{1+b^2x^2}}{\sqrt{1-xx}}\right)$: $\dot{u}=\frac{\dot{x}\sqrt{1+b^2x^2}}{1-x^2}$; which is the Fluxion of the Quantity required: But we are now to get the ſame thing expreſſed in Terms of the Latitude of the Place n: In order thereto, putting the Sine of that Latitude $=s$, we have, by Trigonometry, as $\dot{z}$ $\left(\frac{x\sqrt{1+b^2x^2}}{\sqrt{1-x^2}}\right)$: $-\dot{y}$ $\left(\frac{dx\dot{x}}{\sqrt{1-x^2}}\right)$ :: Radius (1) : s; and conſequently $s\sqrt{1+b^2x^2}=dx$; from which Equation x is found $=\frac{s}{\sqrt{d^2-b^2s^2}}$: Whence $\dot{x}=\frac{d^2\dot{s}}{(d^2-b^2s^2)^{\frac{3}{2}}}$; alſo $1-x^2=\frac{d^2-b^2s^2-s^2}{d^2-b^2s^2}=\frac{d^2-d^2s^2}{d^2-b^2s^2}$ (becauſe $d^2=1+b^2$) and, laſtly, $\sqrt{1+b^2x^2}$ $\left(=\frac{dx}{s}\right)=\frac{d}{\sqrt{d^2-b^2s^2}}$: Which ſeveral Values being ſubſtituted in that of $\dot{u}$, found above, it will become $\left(=\frac{d^2\dot{s}}{(d^2-b^2s^2)^{\frac{3}{2}}}\times\frac{d}{\sqrt{d^2-b^2s^2}}\times\frac{d^2-b^2s^2}{d^2\times\overline{1-ss}}\right)=\frac{d\dot{s}}{\overline{d^2-b^2s^2}\times\overline{1-ss}}$; which reſolved into two Parts, for the more readily finding the Fluent, gives $\dot{u} = \frac{d\dot{s}}{1-s^2} - \frac{db^2\dot{s}}{d^2-b^2s^2}$: Whereof the Fluent being taken, we have

$$u = \begin{cases} 2 \cdot 302585 \text{ \&c. } \times \frac{1}{2} d \times \text{Log. } \frac{1+s}{1-s} \\ - 2 \cdot 302585 \text{ \&c. } \times \frac{1}{2} b \times \text{Log. } \frac{d+bs}{d-bs} \end{cases}$$

But, as 3,14159 *&c.* × 2*d* (the Meaſure of the whole Periphery of the Earth at the Equator, in Parts of the Semi-Axis AD) is to 21600 (the Meaſure of the ſame Periphery in Geographical Miles) ſo is the foreſaid Value of u to

$$\left\{ \begin{array}{l} 3958 \times \text{Log. } \frac{1+s}{1-s} \\ - \frac{3958b}{d} \times \text{Log. } \frac{d+bs}{d-bs} \end{array} \right\} \text{ the correſponding Value}$$

of u, in Geographical Miles, or the Meridional Parts required.

COROLLARY.

466. If the Earth be conſidered as differing but little from a Sphere, d will be nearly $=1$, and conſequently $(\sqrt{d^2-1})$ the Value of b, very ſmall: Therefore, in this Caſe, the latter Part of our Fluent $\left(- \frac{3958b}{d} \times \text{Log. } \frac{d+bs}{d-bs}\right)$ will become nearly $= 3440b^2s$ (becauſe $\text{Log. } \frac{d+bs}{d-bs} = \frac{2bs}{d}) \times \frac{1}{2 \cdot 3025 \text{ \&c.}}$‡. But if the Earth be taken as a perfect Sphere, this laſt Expreſſion will vaniſh, and ſo the Value of u will become barely $=3958$

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‡ *There is a Miſtake in* p. 43. *and* 44. *of my Diſſertations (by forgetting to divide by the Modulus* 2.3025 &c.) *which may from hence be rectify'd.*

$\times$Log. $\frac{1+s}{1-s}$. Which Logarithm, it is eaſy to prove, expreſſes twice the artificial Tangent of half the given Latitude increaſed by 45 Degrees (Radius being Unity.) Wherefore, if the Meridional Parts anſwering to any given Latitude, thus found (from a Table of logarithmic Tangents) when the Earth is conſidered as a perfect Sphere, be denoted by M, it follows that the Meridional Parts anſwering to the ſame Latitude, when the Earth is taken as a Spheroid, will be nearly equal to $M-3440b^2s$: Which, becauſe OP^2 (1) : OA^2 $(1+b^2)$ :: 230 : 231 *, will (by ſubſtituting the Value of b hence ariſing) be reduced to $M-30s$. Whence the following Rule.

* Art. 397.

M

As Radius, to the Sine of the given Latitude, ſo is 30 to a Fourth-Proportional; which ſubtracted from the Meridional Parts when the Earth is taken as a Sphere (found as above) gives the Meridional Parts anſwering to the ſame Latitude, when it is conſidered as an oblate Spheroid.

Thus, for Example, let the given Latitude be 50°: Then, firſt, for the Meridional Parts in the *Sphere*; we muſt, according to the foregoing Preſcript, take the Logarithmic Tangent of 25°+45°, or 70°: Which, by the Table, is found = 0,43893 &c. This multiply'd by the conſtant Multiplicator 7916 (= 2×3958) produces 3475 for the Meridional Parts in the Sphere: Then by the Rule above, it will be as Radius to the Sine of 50°, ſo is 30 to 23; which ſubtracted from 3475, leaves 3452 for the Meridional Parts anſwering to 50° Latitude, in the Spheroid.

PROB. XXX.

467. *To determine the Paths which Shadows of Objects deſcribe, upon the Plane of the Horizon, during the Sun's apparent diurnal Revolution.*

Let CSODT be the Plane of the Horizon, and AV the perpendicular Height of the Object: Then, ſince the Rays, intercepted by the higheſt Point V, would, in the Sun's diurnal Revolution, form a conical Sur-

face

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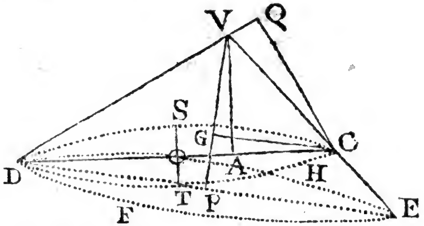
face VDFEH about that Point as a Vertex; whoſe Axis PV produced paſſes thro' the Pole of the World; it is evident that the Path of the Shadow, being the Interſection of the Plane of the Horizon with that Surface, muſt be a Conic Section.



Let its two principal Diameters therefore (when an Ellipſis, that is, when the Sun never deſcends below the Horizon) be CD and ST; alſo let DPE and CG be perpendicular to VP the Axis of the Cone, and CQ perpendicular to DV: Putting the Sine of (QVC) twice the Sun's Declination VEP$=f$; the Sine of (DCV) his greater Meridional Altitude $=g$, and that of the leſſer (CDV) $=b$: Then (*by plane Trig.*) $g : 1$ (AV) $:: 1$ (Radius) : CV $=\frac{1}{g}$; and b (Sine of CDV) : $\frac{1}{g}$ (CV) $:: f$ (Sine of DVC) : DC $=\frac{f}{gb}$: Moreover, 1 (Radius) : $\frac{1}{g}$ (CV) $:: p$ (the Sine of the Comp. Decl. GVC) : GC $=\frac{p}{g}$: And in the very ſame Manner it will be found that DP $=\frac{p}{b}$: But GC $\times$ DP $=$ OS^2 (*vid. Art.* 41.) whence we have ST (2OS) $=\frac{2p}{\sqrt{gb}}$: From which, and the Tranſverſe Axis $\left(DC=\frac{f}{gb}\right)$ the Curve itſelf is given. *Q. E. I.*

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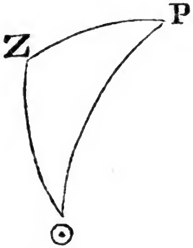
LEMMA.

468. *In any ſpherical Triangle, if Radius be ſuppoſed Unity, the Product of the Sines of any two of the Sides drawn into the Co-ſine of the Angle they include, added to the Product of their Co-ſines, is equal to the Co-ſine of the remaining Side.*

This is demonſtrated by the Writers upon Spherics.

PROB. XXXI.

469. *The Elevation of the Pole and the Declination of the Sun being given, to find at what Time of the Day the Azimuth of the Sun increaſes the ſloweſt.*



It is evident that the Time ſought will be when the Fluxion of the Hour Angle P, bears the greateſt Ratio poſſible to That of the Azimuth Z.

Now the Fluxion of the Angle P is to that of Z, univerſally, as Rad. $\times$ *S.*ZO : *S.* PO $\times$ *Co-ſ.* O (*by Art.* 256. *Caſe* 2.) Conſequently $\frac{S.\,\text{PO}\times Co\text{-}ſ.\,\text{O}}{Rad.\times \text{S. ZO}}$, or $\frac{Co\text{-}ſ.\,\text{O}}{S.\,\text{ZO}}$ is a *Minimum*, in this Caſe, becauſe PO may be conſidered as conſtant.

Let now the Sine of PO be put $=p$, its Co-ſine $=d$, the Co-ſine of PZ $=b$, that of ZO $=x$, and that of O$=y$; then, the Sine of ZO being $=\sqrt{1-x^2}$, we have (*by the Lemma*) $p\sqrt{1-x^2}\times y+dx=b$; whence $y=\frac{b-dx}{p\sqrt{1-x^2}}$, and therefore $\frac{Co\text{-}ſ.\,\text{O}}{S.\,\text{ZO}}\left(=\frac{y}{\sqrt{1-x^2}}\right)$ $=\frac{b-dx}{p\times\overline{1-x^2}}$: Which put into Fluxions, and reduced,

duced, gives $x = \frac{b - \sqrt{b^2 - d^2}}{d}$, for the Sine of the Sun's Altitude at the Time required: Whence the Time itſelf is given.

PROB. XXXII.

470. *To determine the Ratio of the Heat received from the Sun in different Latitudes, during the Time of one whole Day, or any Part thereof.*

Let $p =$ the Sine of the Sun's Polar-Diſtance P☉ (*ſee the laſt Fig.*)
$d =$ its Co-ſine, or the Sine of the Declination.
$b =$ the Sine of the Pole's Elevation.
$c =$ its Co-ſine, or the Sine of PZ.
$z =$ the Angle (P) expreſſing the Time from Noon.
$x =$ its Sine, and $\sqrt{1 - x^2} =$ its Co-ſine.

Then (by the foregoing Lemma) we ſhall have $pc\sqrt{1 - x^2} + bd =$ Co-ſine Z☉ $=$ Sine of the Sun's Altitude

Now, it is known that the Number of Rays falling in any given Particle of Time, upon a given horizontal Plane, is as that Time and the Sine of the Sun's Altitude conjunctly: Therefore the Number of Rays falling in the Time $\dot{z}$, or $\frac{\dot{x}}{\sqrt{1 - xx}}$ (*vid. Art.* 142.) will be defined by $pc\dot{x} + bd\dot{z}$: Whoſe Fluent $pcx + bdz$ is, therefore, as the Heat required.

Where it may be obſerved,

1. That when the Latitude and Declination are of different Kinds, or P☉ is greater than 90 Degrees, the Value of d is to be conſidered as a negative Quantity.

2. That, if the Expreſſion for the Heat found above be divided by the Square of the Sun's Diſtance from the Earth, the Quotient will exhibit the Ratio of the Heat, allowing for the Excentricity of the Earth's Orbit.

COROLLARY I.

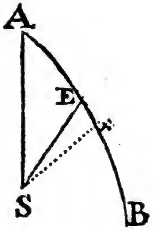
471. If the Place propoſed be at the Equator, the Heat, received in half one diurnal Revolution, will be barely as p; becauſe $b=0$, $c=1$, and $x=1$.

COROLLARY II.

472. But if the Place be at the Pole, then the Heat will be as $d\times\overline{3{,}14159}$ &c. ſince, in this Caſe, $c=0$, $b=1$, and z (= Semi-Circle) $=3{,}14159$ &c.

LEMMA.

473. *The Number of Particles of Light, ejected by the Sun, upon the Earth, in a given Time, is proportional to the Angle deſcribed about his Center in that Time.*

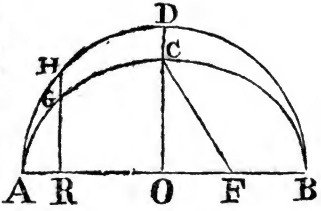


For, let S repreſent the Center of the Sun, AEB the Orbit of the Earth (or That of any other Planet) and let E and r be two Points therein as near as poſſible to each other: Since the Triangle E Sr may be taken as rectilineal, its Area, if the Angle E Sr be ſuppoſed given, or every where the ſame, will be as $SE\times Sr$, or SE^2: And therefore the Time of deſcribing Er (being always as that Area) is alſo explicable by SE^2: But the Intenſity of the Light, or Heat, at the Diſtance of SE is as $\frac{1}{SE^2}$: Therefore the Intenſity compounded with the Time (or the whole Number of Particles received in that Time) will conſequently be as $\frac{1}{SE^2}\times SE^2$ $(=1)$ Which being every where the ſame, the Propoſition is manifeſt.

P R O B. XXXIII.

474. *To determine the Ratio of the Heat received from the Sun at the Equator and either of the Poles, during the Time of one whole Year, or any Part thereof.*

If the Sine of the Sun's Declination be denoted by d and its Co-ſine by p, the Heat received at the Equator, and the Pole, during half one diurnal Revolution of the Sun, wiil be as p and $d \times 3{,}14159$ &c. reſpectively (*by the Corollaries to the preceding Problem.*)



Let the Sun's Longitude, conſidered as variable, be now denoted by z, and its Sine by s; and let f be put for the Sine of the Obliquity of the Ecliptic: Then *(per Spherics)* we ſhall have $d = fs$, and conſequently p $(= \sqrt{1-d^2}) = \sqrt{1-f^2s^2}$: Wherefore, ſeeing the Ratio of Heat in the two Places, for one Half-Day, is that of $\sqrt{1-f^2s^2}$ to $fs \times 3{,}14$ &c. let each of theſe Terms be multiply'd by $\frac{\dot{s}}{\sqrt{1-ss}}$ $(= \dot{z})$ * expreſſing the Quantity of Heat falling upon the Earth in the Time of deſcribing $\dot{z}$ *(ſee the foregoing Lemma)* then the Products $\frac{\dot{s}\sqrt{1-f^2s^2}}{\sqrt{1-s^2}}$, and $3.14f \times \frac{s\dot{s}}{\sqrt{1-s^2}}$ will be the Fluxions of the required Heat, anſwering to $\dot{z}$.

* Art. 142.

But now to exhibit the Fluents hereof, let ACB be an Ellipſis whoſe greater Semi-Axis AO is = Unity, and its Excentricity $FO = f$; and, ſuppoſing ADB to be a Circle deſcribed about the Ellipſis, let the Arch DH expreſs the Sun's Longitude from the Equinoctial Point; whoſe Sine (OR) being $= s$, its Co-ſine RH will be $= \sqrt{1-ss}$.

But, by the Property of the Ellipſis, OD (1) OC : $(\sqrt{1-f^2})$ :: RH $(\sqrt{1-ss})$: RG $= \sqrt{1-ff} \times \sqrt{1-ss}$: Whoſe Fluxion being =

$\frac{\sqrt{1-ff}\times -s\dot{s}}{\sqrt{1-ss}}$, we have $\sqrt{\dot{s}^2+\frac{\overline{1-ff}\times s^2\dot{s}^2}{1-ss}}$ $=\frac{\dot{s}\sqrt{1-f^2s^2}}{\sqrt{1-ss}}=$ the Fluxion of CG. Whence it appears that the Fluent of $\frac{\dot{s}\sqrt{1-f^2s^2}}{\sqrt{1-ss}}$ is truly defined by CG, or CG×AO².

But the Fluent of the other given Fluxion, $3.14f\times\frac{s\dot{s}}{\sqrt{1-ss}}$, will be $=3{,}14f\times\overline{1-\sqrt{1-ss}}=\text{ADB}\times\text{FO}\times\overline{\text{OD}-\text{RH}}$. Therefore the two Fluents, when H and G coincide with A, will be to each other as CA × AO to ADB × FO: Whereof the Antecedent, multiply'd by 4, will be as the Heat received at the Equator during one whole Year; and the Conſequent, multiply'd by 2, as the Heat at the Pole in the ſame Time (becauſe the Sun ſhines at the Pole only two Quarters of the Year.) Hence the required Ratio, of the Heat received at the Equator and Pole, in one whole Year, will be That of CA × AO to DA×FO; or, in Species, as $1-\frac{f^2}{2.2}-\frac{3f^4}{2.2.4.4}-\frac{3\cdot3\cdot5f^6}{2.2.4.4.6.6}$ * &c. to f; which, in Numbers, is as 959 to 396, or as 17 to 7, nearly.

PROB. XXXIV.

475. *To find when that Part of the Equation of Time, ariſing from the Obliquity of the Ecliptic to the Equinoctial, is a* Maximum.

In the right-angled ſpherical Triangle ABC let the Angle A be that made by the Ecliptic AC, and the Equinoctial AB; then the Problem will be, to find when

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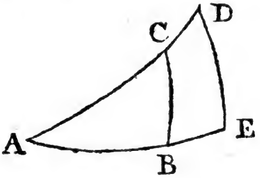
* Art. 434.



when the Difference between the Baſe AB and the Hypothenuſe AC is the greateſt poſſible (the Angle A remaining invariable.) Now (*by Art.* 254.) we have *Co-ſ. BC* : *Sin.* C :: Fluxion of AC : Fluxion of AB: Alſo (*per Spherics*) *Sin.* C : *Co-ſ.* A :: *Rad.* : *Co-ſ.* BC $= \frac{\textit{Co-ſ} \text{ A} \times \textit{Rad.}}{\textit{Sin.} \text{ C}}$: Whence, by multiplying the two firſt Terms of the former Proportion by theſe equal Quantities, reſpectively, we get this new Proportion, *viz.* $\overline{\textit{Co-ſ. BC}}|^2 : \textit{Co-ſ.} \text{ A} \times \textit{Radius}$ :: ſo is the Fluxion of AC to That of AB. But, when AC—AB is a *Maximum*, theſe Fluxions become equal; and conſequently $\overline{\textit{Co-ſ. BC}}|^2 = \textit{Co-ſ. A} \times \textit{Rad.}$ From which Equation BC, and from thence AC, will be known. *Q. E. I.*

The ſame, without Fluxions.

476. It will be (*per Spherics*) *Rad.* : *Co-ſ.* A :: *Tang.* AC : *Tang.* AB; and therefore by Compoſition and Diviſion, *Rad.* + *Co-ſ.* A : *Rad.* — *Co-ſ.* A :: *Tang.* AC + *Tang.* AB : *Tang.* AC — *Tang.* AB :: *Sin.* $\overline{\text{AC+AB}}$: *Sin.* $\overline{\text{AC—AB}}$, by the Theorem mention'd in Problem 8th: From which, by following the Steps there laid down, it appears that, *Radius* + *Co-ſ.* A : *Radius* — *Co ſ.* A :: *Radius* : Sine of $\overline{\text{AC—AB}}$, when a *Maximum*: Whence (AC+AB being then = 90°) both AC and BC will be given.

COROLLARY.

477. Since, *Radius* + *Co-ſ.* A : *Radius* — *Co-ſ.* A :: *Co-tang.* ½ A : *Tang.* ½ A * :: $\overline{\textit{Radius}}|^2 : \overline{\textit{Tang.} \tfrac{1}{2} \text{A}}|^2$;

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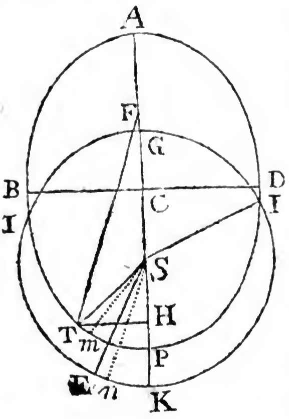
* *Vid. p.* 70. *and* 71. *of my Trigonometry.*

therefore $\overline{Radius}^2 : \overline{Tang.\ \frac{1}{2} A}^2$ :: *Radius* : *Sine* of $\overline{AC - AB}$, Or, *Radius* : *Tang.* $\frac{1}{2}$ A :: *Tang.* $\frac{1}{2}$ **A** : the *Sine* of the greateſt Equation: Which, ſuppoſing the Angle A to be 23° 29′, comes out 2° : 28′ : 34″ : anſwering, in Time, to 9 Minutes : 54 Seconds.

PROB. XXXV.

478. *To determine when the abſolute Equation of Time, ariſing from the Inequality of the Sun's apparent Motion, and the Obliquity of the Ecliptic, conjunctly, is a* Maximum.



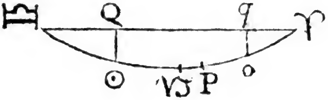
Let ABPD be the Ellipſis in which the Earth revolves about the Sun, in the Focus S; let F be the other Focus, and T the Place of the Earth in its Orbit at the Time required. Moreover, about S, as a Center, let a Circle GEKI be deſcribed, whoſe Diameter GK is a Mean Proportional between the two Axes AP and BD of the Ellipſis; ſo that the Area thereof may be equal to That of the Ellipſis: And, ſuppoſing S*m* to be indefinitely near to ST, let ES*n* be a Sector of the ſaid Circle, equal to the Area TS*m*.

Then, the Time in which the Earth moves thro' the Arch T*m* being to the Time of one intire Revolution, as the Area TS*m*, or ES*n*, is to the whole Ellipſis, or the equal Circle GEKF; and theſe Areas ES*n*, and GEKI being in the Ratio of the Arch E*n* to the whole Periphery GEKI; it is evident that E*n*, or the Angle ESn, will expreſs the Increaſe of the *Mean Longitude*, in the foreſaid Time of deſcribing the Arch Tm: And that this Angle or Increaſe, by reaſon of the Equality of the Areas ESn and TSm, will be to the Angle TSm, expreſſing the correſponding Increaſe of the *True Longitude*, as ST^2 to SE^2. Therefore, if the former be denoted by M, the latter will be repreſented by $\frac{SE^2}{ST^2} \times M$. But now to get a proper Expreſſion for the Value of this Increaſe of the *True Longitude*, in Algebraic Terms; let FT be drawn, and alſo TH, perpendicular to AP: Putting AC $(=CP) = a$, $CB = b$, CS $(=CF) = c$, $ST = z$, and the Co-ſine of (TSP) the Earth's Diſtance from its *Perihelion* (to the Radius 1) $= x$: Then FT being $(= AP - ST) = 2a - z$ (by the Property of the Ellipſis) and $SH = xz$ (*by Trig.*) we have $\overline{FT+ST} \times \overline{FT-ST}$ $(2a \times \overline{2a-2z}) = FS \times 2CH$ $(2 \times 2 \times \overline{c+xz})$ by a known Property of Triangles: From which Equation z (ST) is found $= \frac{a^2-c^2}{a+cx} = \frac{b^2}{a+cx}$: And this Value, with that of ES^2 $(=ab)$ being ſubſtituted in the Increaſe of the True Longitude, found above, we thence get $\frac{a \times \overline{a+cx}|^2}{b^3} \times M$ for the Meaſure of that Increaſe; where M denotes the Increment of the *Mean Motion* correſponding.

This being obtained, let ♎ ♑ ♈ (in the annexed Figure) repreſent the Southern Semi-Circle of the Ecliptic, P the Place of the Perihelion, ♑ the Tropic of Capricorn, ☉ the apparent Place of the Sun in the Ecliptic, and Q☉ his Declination, at the Time required: Then it appears, (*from Art.* 475.) that the Increaſe of the *True Longitude* ♎☉, in an indefinitely ſmall Particle of Time, will be to That of the *Right-Aſcenſion* ♎Q, in the ſame Time, as the Square of the Coſine of Q☉ is to a Rectangle under the Radius and the Co-ſine of the Angle ♎: Therefore, the former,

being

being expreſſed by $\frac{a\times\overline{a+cx}|^2}{b^3}\times M$, the latter is truly repreſented by $\frac{a\times\overline{a+cx}|^2}{b^3}\times M\times\frac{\text{Rad.}\times\text{Co-ſ. ♎}}{\overline{\text{Co-ſ. }\odot Q}|^2}$: Which, in the required Circumſtance, when the propoſed Equation (or the Difference between the Sun's



Mean Motion and *Right Aſcenſion*) is a *Maximum*, muſt conſequently be equal to (M) the correſponding Increaſe of *Mean Motion*; and therefore $\frac{a\times\overline{a+cx}|^2}{b^3}$ $=\frac{\overline{\text{Co-ſ. }\odot Q}|^2}{\text{Rad.}\times\text{Co-ſ. ♎}}$.

But, to obtain the Value of the latter Part of this Equation, alſo, in Algebraic Terms, let the Sine and Co-ſine of (♑P) the Diſtance of the *Perihelion* from ♑, be denoted by m and n reſpectively; then, the Co-ſine of P⊙ being (as above) expreſſed by x, and its Sine by $\sqrt{1-xx}$, we ſhall thence get $nx+m\sqrt{1-xx}=$ Co-ſine of ⊙♑ = Sine of ♎⊙ (*by the Elem. of Trig.*) But (putting the Sine of the Angle ♎ $=p$ and its Co-ſine $=q$) we have (*per Spherics*) Radius (1) : Sine ♎⊙ $(nx+m\sqrt{1-xx})$ :: p : $pnx+pm\sqrt{1-xx}=$ Sine of Q⊙; from whence $\overline{\text{Co-ſ. }Q\odot}|^2$ $=1-\overline{pnx+pm\sqrt{1-xx}}|^2$: Which Value, with That of the Co-ſine of the Angle ♎, being ſubſtituted above, we, at length, get $\frac{a\times\overline{a+cx}|^2}{b^3}=$

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$\frac{\overline{1 - pnx + pm\sqrt{1 - xx}}|^2}{q}$; from which Equation the Value of x may be determined.

The foregoing Equation, it may be obſerved, gives the Time of the *Maximum* which precedes the Winter Solſtice; but if the *Maximum* following that Solſtice be ſought; 'tis but changing the Sign of m, and then you will have $\frac{a \times \overline{a + cx}|^2}{b^3} = \frac{\overline{1 - pnx - pm\sqrt{1 - xx}}|^2}{q}$, anſwering in this Caſe. And from the negative Roots of this, and the preceding, Equation, the Times of the other *Maxima* after, and before, the Summer Solſtice will alſo be obtained. *Q. E. I.*

COROLLARY.

479. It is evident that the Equation of the Earth's Orbit (or that Part of the Equation of Time ariſing from the Inequality of the Sun's apparent Motion) will be a *Maximum*, when the Center of the Earth is in the Interſection of the Ellipſis and the Circle; where the *Mean Motion* and *True Longitude* increaſe with the ſame Celerity.

PROB. XXXVI.

480. *To determine the Law of the Denſity of a Medium and the Curve deſcribed therein, by Means of an uniform Gravity, ſo that the Projectile may, every where, move with the ſame Velocity.*

It appears, from Art. 367. that $\sqrt{\frac{\dot{y}\dot{y}}{\ddot{x}}}$ is a general Expreſſion for the Celerity in the Direction of the Ordinate PBR; whence $\frac{\dot{z}}{\dot{y}} \times \sqrt{\frac{\dot{y}\dot{y}}{\ddot{x}}}$, or its Equal, $\frac{\dot{z}}{\sqrt{\ddot{x}}}$, muſt be the true Meaſure of the abſolute Celerity

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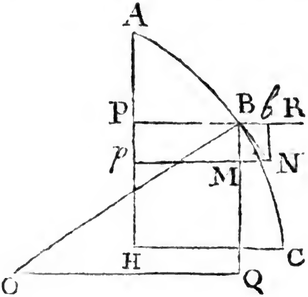
lerity, in the Direction BN: Which being a conſtant Quantity (by Hypotheſis) its Square muſt alſo be con-



ſtant, and ſo, we have $\frac{\dot{z}\dot{z}}{\ddot{x}}=a$; and conſequently $\dot{x}\dot{x}+\dot{y}\dot{y}\ (=\dot{z}\dot{z})=a\ddot{x}$.

But, in order to the Solution of the Equation thus given, make $u:1::\dot{x}:\dot{y}$, or $\dot{x}=u\dot{y}$; then $\ddot{x}=\dot{u}\dot{y}$, and, by Subſtitution, $u^2\dot{y}^2+\dot{y}^2=a\dot{u}\dot{y}$: Hence, $\dot{y}$ being $=\frac{a\dot{u}}{uu+1}$, and $\dot{x}=\frac{au\dot{u}}{uu+1}$, we get $y=a\times$ *Arch*, whoſe
* Art. 142. Tangent is u * (and Secant $\sqrt{1+uu}$); and $x=\frac{1}{2}a\times$
† Art. 126. Hyp. Log. $\overline{1+uu}=a\times$ Hyp. Log. $\sqrt{1+uu}$ †.

Therefore, as the Hyp. Log. of $\sqrt{1+uu}$ is $=\frac{x}{a}$, the Common Logarithm of $\sqrt{1+uu}$ will be $=\frac{\overline{0{,}4342944\,\&c.}\times x}{a}$; and conſequently $y=a\times$ *Arch*, whoſe Radius is Unity, and Log. Secant $\frac{\overline{0.4342944\,\&c.}\times x}{a}$.

Moreover, with reſpect to the Denſity of the Medium; if the abſolute Force of Gravity, in the Direction QB, be

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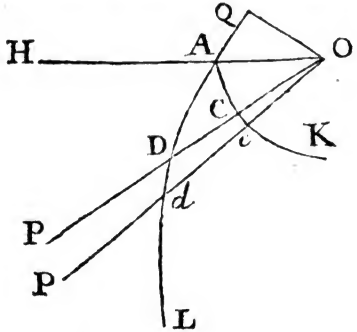
be denoted by Unity, its Efficacy in the Direction BN, whereby the Body is accelerated, will be expreſſed by $\frac{\dot{x}}{\dot{z}}$, or its Equal $\frac{u}{\sqrt{1+uu}}$: Which, as the Velocity is ſuppoſed to remain every where the ſame, muſt alſo expreſs the Force of the Reſiſtance, in the oppoſite Direction, or the true Meaſure of the required Denſity. This, therefore, if M be put for the abſolute Number whoſe Hyperbolical Logarithm is Unity, may be had in Terms of x, and will be $\overline{1-\overline{M}|^{\frac{-2x}{a}}}|^{\frac{1}{2}}$: Becauſe Hyp. Log. $\overline{M}|^{\frac{x}{a}}$ $\left(=\frac{x}{a}\right)$ being $=$ Hyp. Log. $\sqrt{1+uu}$, we have $\sqrt{1+uu}=\overline{M}|^{\frac{x}{a}}$; whence $u=\overline{M^{\frac{2x}{a}}-1}|^{\frac{1}{2}}$, and conſequently $\frac{u}{\sqrt{1+uu}}=\overline{1-\overline{M}|^{\frac{-2x}{a}}}|^{\frac{1}{2}}$. *Q. E. I.*

PROB. XXXVII.

481. *Let a Line, or an inflexible Rod* OP (*conſider'd without regard to Thickneſs*) *be ſuppoſed to revolve about one of its Extremes* O, *as a Center, with a Motion regulated according to any given Law; whilſt a Ring, or Ball, carry'd about with it, and tending to the Center* O *with any given Force, is ſuffer'd to move or ſlide, freely along the ſaid Line or Rod: 'Tis propoſed to determine the Velocity of the Ring, and its Preſſure upon the Rod, in any propoſed Poſition, together with the Nature of the Curve* ADL *deſcribed by means of that compound Motion.*

Let ODP be any Poſition of the revolving Line, and D the correſponding Poſition of the Body: Moreover, ſuppoſing ACK to be the Circumference of a

Circle deſcribed from the Center O, thro' the given Point A, let the Meaſure of the angular Celerity of that Line, in the ſaid Circumference ACK, be repre-



ſented by u; alſo let v denote the Celerity of the Ring at D in the Direction DP; and w the true Meaſure of the centripetal Force: Call OA, a; OD, x; and AC, z; and let the given Values of u and v, at A, be denoted by b and c reſpectively. Then it will be, as $a : x :: u : \left(\frac{ux}{a}\right)$ the paracentric Velocity of the Body, at D; whoſe Square, divided by the Diſtance OD, gives † Art. 211. $\frac{u^2x}{a^2}$, for the true Meaſure of the Centrifugal Force † ariſing from the Revolution of the Rod: From which the centripetal Force w being deducted, the Remainder, $\frac{xu^2}{a^2} - w$, is the true Force whereby the Velocity in the Line OP is accelerated. Therefore (*by Art.* 218.) we have $v\dot{v} = \overline{\frac{xu^2}{a^2} - w} \times \dot{x} = \frac{u^2x\dot{x}}{a^2} - w\dot{x}$.

Moreover, becauſe the Fluxion of the Time is expreſſed either by $\frac{\dot{x}}{v}$ or by $\frac{\dot{z}}{u}$, theſe two Values muſt, there-

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therefore, be equal to each other, and conſequently $v = \frac{u\dot{x}}{\dot{z}}$: From which, and the preceding Equation (when u and w are exhibited in Terms of x or z) the quired Relation of v, x and z will alſo become known— But now, in order to determine the Action of the Rod upon the Ring; let OdP be indefinitely near to ODP, interſecting ADL and ACK in d and c; and put O$d = x + \acute{x}$. Then, becauſe a Body, acted on by no other Force beſides That tending to the Center, about which it revolves, deſcribes Areas proportional to the Times*, and the angular Celerity of a Ray revolving with the Body, is, in that Caſe, as the Square of the Diſtance of the Body from the Center, inverſely (*vid. Art.* 478.) it follows, that, if the Rod was to ceaſe to act upon the Ring, at the Poſition ODP, the angular Celerity at c, would then be $\frac{x^2}{\overline{x+\acute{x}}|^2} \times u$, inſtead of $u + \acute{u}$. Therefore the Exceſs of $u + \acute{u}$ above $\frac{x^2}{\overline{x+\acute{x}}|^2} \times u$, which is $= \acute{u} + \frac{2u\acute{x}}{x} - \frac{3u\acute{x}^2}{x^2}$ &c. is the Increaſe of the ſaid angular Celerity, at the Diſtance OC, ariſing from the Action of the Rod. Therefore it will be, as OC (a) : OD (x) :: the ſaid Increaſe to $\left(\frac{x\acute{u}}{a} + \frac{2u\acute{x}}{a} - \frac{3u\acute{x}^2}{ax}\ \&c.\right)$ the Alteration of the Ring's paracentric Velocity, ariſing from the ſame Cauſe. Which, divided by $\left(\frac{\acute{x}}{v}\right)$ the Time wherein *It* is produced, gives $\frac{xv\acute{u}}{a\acute{x}} + \frac{2uv}{a} -$

* Art. 224.

$\frac{3uv\overset{\prime}{x}}{ax}$ *&c.* for the Meaſure of the Force, by which *It* is produced. From whence, by ſubſtituting $\frac{\dot{u}}{\dot{x}}$ in the Room of $\frac{\overset{\prime}{u}}{\overset{\prime}{x}}$, and neglecting all the Terms after the two
* Art. 134. firſt (in order to have the limiting Ratio *) we get $\frac{xv\dot{u}}{a\dot{x}} + \frac{2uv}{a}$. Therefore it will be, as $\frac{xv\dot{u}}{a\dot{x}} + \frac{2uv}{a}$ to
† Art. 211. $\frac{bb}{a}$ †, or as $\frac{xv\dot{u}}{bb\dot{x}} + \frac{2uv}{bb}$ to Unity, ſo is the Action of the Rod upon the Ring, to the (given) Centrifugal Force at A (or the Force that would retain a Body in the Circle ACK, with the Velocity *b*.) *Q. E. I.*

COROLLARY I.

482. If the angular Motion be uniform, the Equations found above, will become $v\dot{v} = \frac{b^2x\dot{x}}{a^2} - w\dot{x}$, and $v = \frac{b\dot{x}}{\dot{z}}$. From the latter of which, by taking the Fluxion, we have $\dot{v} = \frac{b\ddot{x}}{\dot{z}}$; whence (by Subſtitution) $\frac{b^2\dot{x}\ddot{x}}{\dot{z}\dot{z}} = \frac{b^2x\dot{x}}{aa} - w\dot{x}$, and conſequently $\ddot{x} - \frac{x\dot{z}^2}{a^2} = -\frac{w\dot{z}^2}{b^2}$; from the Solution of which, the Relation of x and z will be given. And then, the Value of v $\left(\frac{b\dot{x}}{\dot{z}}\right)$ being alſo known, the Action upon the Rod, which in this Caſe is barely $= \frac{2bv}{a}$ $\left(= \frac{2b^2\dot{x}}{a\dot{z}}\right)$ will be given likewiſe,

being to $\left(\frac{bb}{a}\right)$ the centrifugal Force in the Circle ACK, as $\frac{2\dot{x}}{\dot{z}}$ to Unity.

COROLLARY II.

483. But if the Angular Celerity be proportional to any Power (x^m) of the Diſtance, and the Centripetal Force w be, alſo, ſuppoſed to vary according to ſome Power (x^n) of the ſame Diſtance: Then, putting p to denote the Centripetal, and q the Centrifugal, Force, at the given Point A, the Value of w will, here, be expounded by $\frac{x^n}{a^n}\times p$, and That of u by $\frac{x^m}{a^m}\times b$: And therefore, the paracentric Velocity of the Ring at D being $=$ $\frac{x^m}{a^m}\times b\times\frac{x}{a}\left(=\frac{bx^{m+1}}{a^{m+1}}\right)$ it will be as $\frac{bb}{a}:\frac{b^2x^{2m+2}}{xa^{2m+2}}$ $::q:\frac{x^{2m+1}}{a^{2m+1}}\times q$, the Centrifugal Force at D*. Hence *Art. 211.

$v\dot{v}=\frac{qx^{2m+1}\dot{x}}{a^{2m+1}}-\frac{px^n\dot{x}}{a^n}$; whereof the (corrected) Fluent is $\frac{1}{2}vv-\frac{1}{2}cc=\frac{qx^{2m+2}}{\overline{2m+2}\times a^{2m+1}}-\frac{px^{n+1}}{\overline{n+1}\times a^n}-\frac{qa}{2m+2}$ $+\frac{pa}{n+1}$: From whence v is found $=$

$$\sqrt{cc-\frac{qa}{m+1}+\frac{2pa}{n+1}+\frac{qx^{2m+2}}{\overline{m+1}\,.\,a^{2m+1}}-\frac{2px^{n+1}}{\overline{n+1}\,.\,a^n}}$$

and $\dot{z}\left(=\frac{u\dot{x}}{v}=\frac{bx^m\dot{x}}{a^m v}\right)=$

$$\frac{bx^m\dot{x}}{a^m\sqrt{cc-\frac{qa}{m+1}+\frac{2pa}{n+1}+\frac{qx^{2m+2}}{\overline{m+1}\,.\,a^{2m+1}}-\frac{2px^{n+1}}{\overline{n+1}\,.\,a^n}}}.$$

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Moreover, by ſubſtituting for u, and its Fluxion, we get $\frac{x\dot{v}\dot{u}}{a\dot{x}}+\frac{2uv}{a}=\overline{m+2}\times\frac{bx^m v}{a^{m+1}}$, expreſſing the Action of the Rod upon the Ring: Which, therefore, when m is expounded by -2, will intirely vaniſh: And, in that Caſe, $\dot{z}$ will become $=$

$$\frac{a^2b\dot{x}}{x\sqrt{\overline{cc+qa+\frac{2pa}{n+1}}\times x^2-qa^3-\frac{2px^{n+3}}{\overline{n+1}\cdot a^n}}};$$

expreſſing the Nature of the Trajectory deſcribed by means of a Centripetal Force, varying according to any Power (x^n) of the Diſtance. But this Equation will be render'd ſomewhat more commodious, by ſubſtituting the Values of b and c: For, if OQ (perpendicular to the Tangent at A) be denoted by h, it will be, h : $\sqrt{a^2-h^2}$ (AQ) :: b (the Celerity in the Direction AC)

* Art. 35. to $c=\frac{b\sqrt{a^2-h^2}}{h}=$ the Celerity in the Direction AH *.

† Art. 211. Therefore, b being $=\sqrt{aq}$ †, we have $c^2=\frac{a^3q}{hh}-aq$;

and $$\dot{z}=\frac{a^2\dot{x}}{x\sqrt{\overline{\frac{aa}{hh}+\frac{2p}{\overline{n+1}\cdot q}}\times x^2-a^2-\frac{2px^{n+3}}{\overline{n+1}\cdot qa^{n+1}}}}:$$

Which Equation is the ſame, in effect, with that given in *Art.* 242. by a different Method.

COROLLARY III.

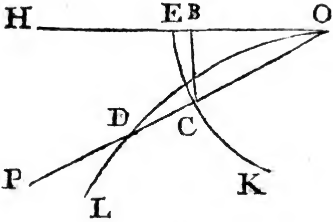
484. If the Angular Celerity be ſuppoſed uniform, and the Ring to have no other Motion along the Rod than what it acquires from its Centrifugal Force; then c, m and p being all of them equal to Nothing, $\dot{z}$ will here become, barely $=\frac{b\dot{x}}{\sqrt{-qa+\frac{qx^2}{a}}}=\frac{a\dot{x}}{\sqrt{x^2-a^2}}$: And

therefore $z = a \times \text{Hyp. Log.} \frac{x + \sqrt{xx - aa}}{a}$. Hence if the Number whoſe Hyp. Log. is $\frac{z}{a}$ be denoted by N, we ſhall have $\frac{x + \sqrt{xx - aa}}{a} = N$: From which x is found $= a \times \overline{\frac{N}{2} + \frac{1}{2N}}$; whence $\dot{x}$ is, alſo, had $= \frac{a\dot{N}}{2} - \frac{a\dot{N}}{2N^2} = \frac{N\dot{z}}{2} - \frac{\dot{z}}{2N}$ (becauſe $\frac{\dot{N}}{N} = \frac{\dot{z}}{a}$). Therefore, it will be (*by Corol.* 1.) as Unity is to $\frac{N}{2} - \frac{1}{2N}$, ſo is the Angular Velocity (b) in the Arch ACK to the Velocity with which the Body recedes from the Center of Motion: And ſo, likewiſe, is the Centrifugal Force in that Arch to half the Preſſure upon the Rod—By taking $z =$ the whole Periphery, or $\frac{z}{a} = 2 \times \overline{3.145}$ *&c.* N will come out $= 535.5$, and $x = 267.7 \times a$: From whence it appears that the Diſtance of the Ring from the Center at the End of one intire Revolution will be almoſt 268 times as great as at firſt.

Corollary IV.

485. If a Body be ſuppoſed to deſcend from the Point O, (*ſee the next Fig.*) by the Force of its own Gravity, along an inclin'd Plane OCP; whilſt the Plane itſelf moves uniformly about that Point, from an horizontal Poſition OEH; then the Place, and the Preſſure of the Body upon the Plane, in any given Poſition OCP, may, alſo, be derived from the Equations in *Corollary* 1. For let CB (perpendicular to OH) be put $= y$; and let the Ratio of the Centrifugal Force in the Circle ECK, to the Force of Gravity (*given by Art.* 217.) be as r to *Unity*: Then, as the Meaſure of the former Force is expreſſed by $\frac{bb}{a}$,

That of the latter muſt be repreſented by $\frac{bb}{ra}$; and, conſequently, its Efficacy in the Direction PO, by $\frac{bby}{raa}$ $\left(=\frac{bb}{ra}\times\frac{CB}{OC}\right)$: Which Value being ſubſtituted for $-w$, in the aforeſaid Corollary, we have $\ddot{x} - \frac{x\dot{z}^2}{aa} = \frac{y\dot{z}^2}{raa}$. But now, in order to the Solution of this



Equation, put the Radius OC $(a) = 1$ (that the Operation may be as ſimple as poſſible) alſo, inſtead of y,

* Art. 425. let its Equal $z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5}$ * &c. be ſubſtituted, and let x be aſſumed $= Az^3 + Bz^5 + Cz^7 + Dz^9$ &c.

Then, by proceeding as is taught in *Art.* 267. the Value of x will come out $= \frac{1}{r}$ into $\frac{z^3}{2.3} + \frac{z^7}{2.3.4.5.6.7} + \frac{z^{11}}{2.3.4.5.6.7.8.9.10.11} +$ &c. Whence the Velocity $\left(\frac{b\dot{x}}{\dot{z}}\right)$ in the Plane, is, alſo, found $= \frac{b}{r}$ into $\frac{z^2}{2} + \frac{z^6}{2.3.4.5.6}$ &c. Which, therefore, is to (*b*) the angular Velocity of the Plane, in the Arch ECK, as $\frac{z^2}{z} + \frac{z^6}{2.3.4.5.6} +$ &c. to r. Moreover, the Centrifugal Force in the ſaid Arch being denoted by r (the Force of Gravity being Unity) it will likewiſe be (by the above-mention'd *Corol.*) as $1 : \frac{2\dot{x}}{\dot{z}} :: r : \left(\frac{2r\dot{x}}{\dot{z}} = \right)$ $z^2 + \frac{z^6}{3.4.5.6} + \frac{z^{10}}{3.4.5.6.7.8.9.10} +$ &c. $=$ the Force ſufficient to keep the Body upon the Plane. But the Force of Gravity in a Direction perpendicular to the Plane (the Weight of the Body being repreſented by *Unity*) is $\frac{OB}{OC} = 1 - \frac{z^2}{2} + \frac{z^4}{2.3.4}$ * &c. From * Art. 425. which deducting the Quantity laſt found, there reſts $1 - \frac{3z^2}{2} + \frac{z^4}{2.3.4} - \frac{3z^6}{2.3.4.5.6}$ &c. for the true Preſſure of the Body upon the Plane. By putting of Which equal to Nothing, z^2 will be found $= 0.67715$; anſwering to an Angle (EOC) of 47° : 9′: Which Angle is therefore the Inclination, when the Force of Gravity is no longer ſufficient to keep the Body upon the Plane.

Though the Value of x, given above, is found by an Infinite Series, yet the Sum of that Series is eaſily exhibited by the Meaſures of Angles and Ratios. For, putting N to denote the Number whoſe hyperbolical Logarithm is z,

we have * $$\left\{\begin{array}{l} 1 + z + \frac{z^2}{2} + \frac{z^3}{2.3} + \frac{z^4}{2.3.4} \text{ \&c. } = N \\ 1 - z + \frac{z^2}{z} - \frac{z^3}{2.3} + \frac{z^4}{2.3.4} \text{ \&c. } = \frac{1}{N}: \end{array}\right.$$ * Art. 424.

Half the Difference of which two Equations is $z + \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} + \frac{z^7}{2.3.4.5.6.7}$ &c. $= \frac{N}{2} - \frac{1}{2N}$:

From which taking $z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} - \frac{z^7}{2.3.4.5.6.7}$ *&c.* $= y$; and dividing the Remainder by $2r$, there reſults $\left(\frac{1}{r} \times \overline{\frac{z^3}{2.3} + \frac{z^7}{2.3.4.5.6.7} \,\&c.}\right) \frac{1}{2r} \times \overline{\frac{N}{2} - \frac{1}{2N} - y}$, for the true Value of x. Which, if required, may be expreſſed independent of r; by putting d for the Diſtance thro' which a Body, freely, deſcends in the firſt Second of Time, and taking b to denote the Velocity of the Plane, *(per Second)* in the Arch ECK: For then, the Ratio of the Centrifugal Force, in the ſaid Arch, to the Force of Gravity (or

* Art. 211. That of r to 1) being as $\frac{bb}{1}$ $\left(= \frac{bb}{OC}\right)$ to $2d$ *, we ſhall have $r = \frac{bb}{2d}$, and conſequently $x = \frac{d}{bb} \times \overline{\frac{N}{2} - \frac{1}{2N} - y}$.

By Computations, not very unlike Thoſe above, the Motion of the Moon's *Apogee*, and the principal Equations of the Lunar Orbit may be exhibited; by means of proper Approximations, derived from the general Equations in *Art.* 481. But this is a Conſideration that would require a Volume of itſelf, to treat it, from firſt Principles, with all the Attention and Perſpicuity ſuitable to the Importance of the Subject. I ſhall conclude this Work with the following ſhort Table of *Hyperbolical Logarithms*, drawn up and communicated by my ingenious Friend Mr. *John Turner*: Whereof the Uſe, in finding Fluents, will ſufficiently appear from the foregoing Pages. In the ſaid Table we have given the Hyperbolical Logarithms of every whole Number and hundredth Part of an Unit, from 1 to 10 (which Form is beſt adapted to the Purpoſes above-mention'd) by Help whereof, and the following Obſervations, the Hyperbolical

perbolical Logarithm of any Number, not exceeding ſeven Places of Figures, may be found with very little Trouble.

1°. *If the Number given be between* 1 *and* 10 *(ſo as to fall within the Limits of the Table.)*

Then take from it the next inferior Number in the Table, and divide the Remainder by the ſaid inferior Number increaſed by half the Remainder; and let the Quotient be added to the Logarithm of the ſaid inferior Number, the Sum will be the Logarithm ſought.

Thus, let the Hyperbolical Logarithm of 3.45678 be required; then the Operation will ſtand thus: 3.45339).005678(.0016442: Which added to 1.2383742, the Log. of 3.45, gives 1.2400184 for the Logarithm ſought.

2°. *When the Number propoſed exceeds* 10.

Find the Logarithm thereof, ſuppoſing all the Figures after the Firſt to be Decimals; then to the Logarithm, ſo found, let 2.3025851, 4.6051702, or 6.9077553 *&c.* be added, according as the whole Number conſiſts of 2, 3, or 4 *&c.* Places: The Sum will be the Logarithm ſought.

Thus, the Hyperbolical Logarithm of 345.678 will be found to be 5.8451886: For That of 3.45678 being 1.2400184; the ſame, added to 4.6051702, gives the very Quantity above exhibited. The Reaſon of which, as well as of the Operation in the preceding Caſe, is evident from the Nature and Conſtruction of Logarithms.

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N	Logarithm	N	Logarithm	N	Logarithm
1.01	.0099503	1.34	.2926696	1.67	.5128236
1.02	.0198026	1.35	.3001045	1.68	.5187937
1.03	.0295588	1.36	.3074846	1.69	.5247285
1.04	.0392207	1.37	.3148107	1.70	.5306282
1.05	.0487902	1.38	.3220834	1.71	.5364933
1.06	.0582689	1.39	.3293037	1.72	.5423242
1.07	.0676586	1.40	.3364722	1.73	.5481214
1.08	.0769610	1.41	.3435897	1.74	.5538851
1.09	.0861777	1.42	.3506568	1.75	.5596157
1.10	.0953102	1.43	.3576744	1.76	.5653138
1.11	.1043600	1.44	.3646431	1.77	.5709795
1.12	.1133287	1.45	.3715635	1.78	.5766133
1.13	.1222176	1.46	.3784364	1.79	.5822156
1.14	.1310283	1.47	.3852624	1.80	.5877866
1.15	.1397619	1.48	.3920420	1.81	.5933268
1.16	.1484200	1.49	.3987761	1.82	.5988365
1.17	.1570037	1.50	.4054651	1.83	.6043159
1.18	.1655144	1.51	.4121096	1.84	.6097655
1.19	.1739533	1.52	.4187103	1.85	.6151856
1.20	.1823215	1.53	.4252677	1.86	.6205764
1.21	.1906203	1.54	.4317824	1.87	.6259384
1.22	.1988508	1.55	.4382549	1.88	.6312717
1.23	.2070141	1.56	.4446858	1.89	.6365768
1.24	.2151113	1.57	.4510756	1.90	.6418538
1.25	.2231435	1.58	.4574248	1.91	.6471032
1.26	.2311117	1.59	.4637340	1.92	.6523251
1.27	.2390169	1.60	.4700036	1.93	.6575200
1.28	.2468600	1.61	.4762341	1.94	.6626879
1.29	.2546422	1.62	.4824261	1.95	.6678293
1.30	.2623642	1.63	.4885800	1.96	.6729444
1.31	.2700271	1.64	.4946962	1.97	.6780335
1.32	.2776317	1.65	.5007752	1.98	.6830968
1.33	.2851789	1.66	.5068175	1.99	.6881346
1.34	.2926696	1.67	.5128236	2.00	.6931472

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N	Logarithm	N	Logarithm	N	Logarithm
2.01	.6981347	2.34	.8501509	2.67	.9820784
2.02	.7030974	2.35	.8544153	2.68	.9858167
2.03	.7080357	2.36	.8586616	2.69	.9895411
2.04	.7129497	2.37	.8628899	2.70	.9932517
2.05	.7178397	2.38	.8671004	2.71	.9969486
2.06	.7227059	2.39	.8712933	2.72	1.0006318
2.07	.7275485	2.40	.8754687	2.73	1.0043015
2.08	.7323678	2.41	.8796267	2.74	1.0079579
2.09	.7371640	2.42	.8837675	2.75	1.0116008
2.10	.7419373	2.43	.8878912	2.76	1.0152306
2.11	.7466879	2.44	.8919980	2.77	1.0188473
2.12	.7514160	2.45	.8960880	2.78	1.0224509
2.13	.7561219	2.46	.9001613	2.79	1.0260415
2.14	.7608058	2.47	.9042181	2.80	1.0296194
2.15	.7654678	2.48	.9082585	2.81	1.0331844
2.16	.7701082	2.49	.9122826	2.82	1.0367368
2.17	.7747271	2.50	.9162907	2.83	1.0402766
2.18	.7793248	2.51	.9202827	2.84	1.0438040
2.19	.7839015	2.52	.9242589	2.85	1.0473189
2.20	.7884573	2.53	.9282193	2.86	1.0508216
2.21	.7929925	2.54	.9321640	2.87	1.0543120
2.22	.7275071	2.55	.9360933	2.88	1.0577902
2.23	.8020015	2.56	.9400072	2.89	1.0612564
2.24	.8064758	2.57	.9439058	2.90	1.0647107
2.25	.8109302	2.58	.9477893	2.91	1.0681530
2.26	.8153648	2.59	.9516578	2.92	1.0715836
2.27	.8197798	2.60	.9555114	2.93	1.0750024
2.28	.8241754	2.61	.9593502	2.94	1.0784095
2.29	.8285518	2.62	.9631743	2.95	1.0818051
2.30	.8329091	2.63	.9669838	2.96	1.0851892
2.31	.8372475	2.64	.9707789	2.97	1.0885619
2.32	.8415671	2.65	.9745596	2.98	1.0919233
2.33	.8458682	2.66	.9783261	2.99	1.0952733
2.34	.8501509	2.67	.9820784	3.00	1.0986123

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N	Logarithm	N	Logarithm	N	Logarithm
3.01	1.1019400	3.34	1.2059707	3.67	1.3001916
3.02	1.1052568	3.35	1.2089603	3.68	1.3029127
3.03	1.1085626	3.36	1.2119409	3.69	1.3056264
3.04	1.1118575	3.37	1.2149127	3.70	1.3083328
3.05	1.1151415	3.38	1.2178757	3.71	1.3110318
3.06	1.1184149	3.39	1.2208299	3.72	1.3137236
3.07	1.1216775	3.40	1.2237754	3.73	1.3164082
3.08	1.1249295	3.41	1.2267122	3.74	1.3190856
3.09	1.1281710	3.42	1.2296405	3.75	1.3217558
3.10	1.1314021	3.43	1.2325605	3.76	1.3244189
3.11	1.1346227	3.44	1.2354714	3.77	1.3270749
3.12	1.1378330	3.45	1.2383742	3.78	1.3297240
3.13	1.1410330	3.46	1.2412685	3.79	1.3323660
3.14	1.1442227	3.47	1.2441545	3.80	1.3350010
3.15	1.1474024	3.48	1.2470322	3.81	1.3376291
3.16	1.1505720	3.49	1.2499017	3.82	1.3402504
3.17	1.1537315	3.50	1.2527629	3.83	1.3428648
3.18	1.1568811	3.51	1.2556160	3.84	1.3454723
3.19	1.1600209	3.52	1.2584609	3.85	1.3480731
3.20	1.1631508	3.53	1.2612978	3.86	1.3506671
3.21	1.1662709	3.54	1.2641266	3.87	1.3532544
3.22	1.1693813	3.55	1.2669475	3.88	1.3558351
3.23	1.1724821	3.56	1.2697605	3.89	1.3584091
3.24	1.1755733	3.57	1.2725655	3.90	1.3609765
3.25	1.1786549	3.58	1.2753627	3.91	1.3635373
3.26	1.1817271	3.59	1.2781521	3.92	1.3660916
3.27	1.1847899	3.60	1.2809338	3.93	1.3686394
3.28	1.1878434	3.61	1.2837077	3.94	1.3711807
3.29	1.1908875	3.62	1.2864740	3.95	1.3737156
3.30	1.1939224	3.63	1.2892326	3.96	1.3762440
3.31	1.1969481	3.64	1.2919836	3.97	1.3787661
3.32	1.1999647	3.65	1.2947271	3.98	1.3812818
3.33	1.2029722	3.66	1.2974631	3.99	1.3837912
3.34	1.2059707	3.67	1.3001916	4.00	1.3862943

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N	Logarithm	N	Logarithm	N	Logarithm
4.01	1.3887912	4.34	1.4678743	4.67	1.5411590
4.02	1.3912818	4.35	1.4701758	4.68	1.5432981
4.03	1.3937663	4.36	1.4724720	4.69	1.5454325
4.04	1.3962446	4.37	1.4747630	4.70	1.5475625
4.05	1.3987168	4.38	1.4770487	4.71	1.5496879
4.06	1.4011829	4.39	2.4793292	4.72	1.5518087
4.07	1.4036429	4.40	1.4816045	4.73	1.5539252
4.08	1.4060969	4.41	1.4838746	4.74	1.5560371
4.09	1.4085449	4.42	1.4861396	4.75	1.5581446
4.10	1.4109869	4.43	1.4883995	4.76	1.5602476
4.11	1.4134230	4.44	1.4906543	4.77	1.5623462
4.12	1.4158531	4.45	1.4929040	4.78	1.5644405
4.13	1.4182774	4.46	1.4951487	4.79	1.5665304
4.14	1.4206957	4.47	1.4973883	4.80	1.5686159
4.15	1.4231083	4.48	1.4996230	4.81	1.5706971
4.16	1.4255150	4.49	1.5018527	4.82	1.5727739
4.17	1.4279160	4.50	1.5040774	4.83	1.5748464
4.18	1.4303112	4.51	1.5062971	4.84	1.5769147
4.19	1.4327007	4.52	1.5085119	4.85	1.5789787
4.20	1.4350845	4.53	1.5107219	4.86	1.5810384
4.21	1.4374626	4.54	1.5129269	4.87	1.5830939
4.22	1.4398351	4.55	1.5151272	4.88	1.5851452
4.23	1.4422020	4.56	1.5173226	4.89	1.5871923
4.24	1.4445632	4.57	1.5195132	4.90	1.5892352
4.25	1.4469189	4.58	1.5216990	4.91	1.5912739
4.26	1.4492691	4.59	1.5238800	4.92	1.5933085
4.27	1.4516138	4.60	1.5260563	4.93	1.5953389
4.28	1.4539530	4.61	1.5282278	4.94	1.5973653
4.29	1.4562867	4.62	1.5303947	4.95	1.5993875
4.30	1.4586149	4.63	1.5325568	4.96	1.6014057
4.31	1.4609379	4.64	1.5347143	4.97	1.6034198
4.32	1.4632553	4.65	1.5368672	4.98	1.6054298
4.33	1.4655675	4.66	1.5390154	4.99	1.6074358
4.34	1.4678743	4.67	1.5411590	5.00	1.6094379

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N	Logarithm	N	Logarithm	N	Logarithm
5.01	1.6114359	5.34	1.6752256	5.67	1.7351891
5.02	1.6134300	5.35	1.6770965	5.68	1.7369512
5.03	1.6154200	5.36	1.6789639	5.69	1.7387102
5.04	1.6174060	5.37	1.6808278	5.70	1.7404661
5.05	1.6193882	5.38	1.6826882	5.71	1.7422189
5.06	1.6213664	5.39	1.6845453	5.72	1.7439687
5.07	1.6233408	5.40	1.6863989	5.73	1.7457155
5.08	1.6253112	5.41	1.6882491	5.74	1.7474591
5.09	1.6272778	5.42	1.6900958	5.75	1.7491998
5.10	1.6292405	5.43	1.6919391	5.76	1.7509374
5.11	1.6311994	5.44	1.6937790	5.77	1.7526720
5.12	1.6331544	5.45	1.6956155	5.78	1.7544036
5.13	1.6351056	5.46	1.6974487	5.79	1.7561323
5.14	1.6370530	5.47	1.6992786	5.80	1.7578579
5.15	1.6389967	5.48	1.7011051	5.81	1.7595805
5.16	1.6409365	5.49	1.7029282	5.82	1.7613002
5.17	1.6428726	5.50	1.7047481	5.83	1.7630170
5.18	1.6448050	5.51	1.7065646	5.84	1.7647308
5.19	1.6467336	5.52	1.7083778	5.85	1.7664416
5.20	1.6486586	5.53	1.7101878	5.86	1.7681496
5.21	1.6505798	5.54	1.7119944	5.87	1.7698546
5.22	1.6524974	5.55	1.7137979	5.88	1.7715567
5.23	1.6544112	5.56	1.7155981	5.89	1.7732559
5.24	1.6563214	5.57	1.7173950	5.90	1.7749523
5.25	1.6582280	5.58	1.7191887	5.91	1.7766458
5.26	1.6601310	5.59	1.7209792	5.92	1.7783364
5.27	1.6620303	5.60	1.7227666	5.93	1.7800242
5.28	1.6639260	5.61	1.7245507	5.94	1.7817091
5.29	1.6658182	5.62	1.7263316	5.95	1.7833912
5.30	1.6677068	5.63	1.7281094	5.96	1.7850704
5.31	1.6695918	5.64	1.7298840	5.97	1.7867469
5.32	1.6714733	5.65	1.7316555	5.98	1.7884205
5.33	1.6733512	5.66	1.7334238	5.99	1.7900914
5.34	1.6752256	5.67	1.7351891	6.00	1.7917594

N	Logarithm	N	Logarithm	N	Logarithm
6.01	1.7934247	6.34	1.8468787	6.67	1.8976198
6.02	1.7950872	6.35	1.8484547	6.68	1.8991179
6.03	1.7967470	6.36	1.8500283	6.69	1.9006138
6.04	1.7984040	6.37	1.8515994	6.70	1.9021075
6.05	1.8000582	6 38	1.8531680	6.71	1.9035989
6.06	1.8017098	6.39	1.8547342	6.72	1.9050881
6.07	1.8033586	6.40	1.8562979	6.73	1.9065751
6.08	1.8050047	6.41	1.8578592	6.74	1.9080600
6.09	1.8066481	6.42	1.8594181	6.75	1.9095425
6.10	1.8082887	6.43	1.8609745	6.76	1.9110228
6.11	1.8099267	6.44	1.8625285	6.77	1.9125011
6.12	1.8115621	6.45	1.8640801	6.78	1.9139771
6.13	1.8131947	6.46	1.8656293	6.79	1.9154509
6.14	1.8148247	6.47	1.8671761	6.80	1.9169226
6.15	1.8164520	6.48	1.8687205	6.81	1.9183921
6.16	1.8180767	6.49	1.8702625	6.82	1.9198594
6.17	1.8196988	6.50	1.8718021	6.83	1.9213247
6.18	1.8213182	6.51	1.8733394	6.84	1.9227877
6.19	1.8229351	6.52	1.8748743	6.85	1.9242486
6.20	1.8245493	6.53	1.8764069	6.86	1.9257074
6.21	1.8261608	6.54	1.8779371	6.87	1.9271641
6.22	1.8277699	6.55	1.8794650	6.88	1.9286186
6.23	1.8293763	6.56	1.8809906	6.89	1.9300710
6.24	1.8309801	6.57	1.8825138	6.90	1.9315214
6.25	1.8325814	6.58	1.8840347	6.91	1.9329696
6.26	1.8341801	6.59	1.8855533	6.92	1.9344157
6.27	1.8357763	6.60	1.8870696	6.93	1.9358598
6.28	1.8373699	6 61	1.8885837	6.94	1.9373017
6.29	1.8389610	6.62	1.8900954	6.95	1.9387416
6.30	1.8405496	6.63	1.8916048	6.96	1.9401794
6.31	1.8421356	6.64	1.8931119	6.97	1.9416152
6.32	1.8437191	6.65	1.8946168	6.98	1.9430489
6.33	1.8453002	6.66	1.8961194	6.99	1.9444805
6.34	1.8468787	6.67	1.8976198	7.00	1.9459101

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N	Logarithm	N	Logarithm	N	Logarithm
7.01	1.9473376	7.34	1.9933387	7.67	2.0373166
7.02	1.9487632	7.35	1.9947002	7.68	2.0386195
7.03	1.9501866	7.36	1.9960599	7.69	2.0399207
7.04	1.9516080	7.37	1.9974177	7.70	2.0412203
7.05	1.9530275	7.38	1.9987736	7.71	2.0425181
7.06	1.9544449	7.39	2.0001278	7.72	2.0438143
7.07	1.9558604	7.40	2.0014800	7.73	2.0451088
7.08	1.9572739	7.41	2.0028305	7 74	2.0464016
7.09	1.9586853	7 42	2.0041790	7.75	2.0476928
7.10	1.9600947	7 43	2.0055258	7.76	2 0489823
7.11	1.9615022	7.44	2.0068708	7.77	2.0502701
7.12	1.9629077	7.45	2.0082140	7.78	2.0515563
7.13	1.9643112	7.46	2.0095553	7.79	2.0528408
7.14	1.9657127	7.47	2.0108949	7.80	2.0541237
7.15	1.9671123	7.48	2.0122327	7.81	2.0554049
7.16	1.9685099	7.49	2.0135687	7.82	2.0566845
7.17	1.9699056	7.50	2.0149030	7.83	2.0579624
7.18	1.9712993	7.51	2.0162354	7.84	2.0592388
7 19	1.9726911	7.52	2.0175661	7.85	2.0605135
7.20	1.9740810	7.53	2.0188950	7.86	2.0617866
7.21	1.9754689	7.54	2.0202221	7.87	2.0630580
7.22	1.9768549	7.55	2.0215475	7.88	2.0643278
7.23	1.9782390	7.56	2 0228711	7.89	2.0655961
7.24	1.9796212	7.57	2.0241929	7.90	2.0668627
7.25	1.9810014	7.58	2.0255131	7.91	2.0681277
7.26	1.9823798	7.59	2.0268315	7.92	2.0693911
7.27	1.9837562	7.60	2.0281482	7.93	2.0706530
7 28	1.9851308	7.61	2.0294631	7.94	2.0719132
7.29	1.9865035	7.62	2.0307763	7.95	2.0731719
7.30	1.9878743	7.63	2.0320878	7.96	2.0744290
7.31	1.9892432	7.64	2.0333976	7.97	2.0756845
7.32	1.9906103	7.65	2.0347056	7.98	2.0769384
7.33	1.9919754	7.66	2.0360119	7.99	2.0781907
7.34	1.9933387	7.67	2.0373166	8.00	2 0794415

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N	Logarithm	N	Logarithm	N	Logarithm
8.01	2.0806907	8.34	2.1210632	8.67	2.1598687
8.02	2.0819384	8.35	2.1222615	8.68	2.1610215
8.03	2.0831845	8.36	2.1234584	8.69	2.1621729
8.04	2.0844290	8.37	2.1246539	8.70	2.1633230
8.05	2.0856720	8.38	2.1258479	8.71	2.1644718
8.06	2.0869135	8.39	2.1270405	8.72	2.1656192
8.07	2.0881534	8.40	2.1282317	8.73	2.1667653
8.08	2.0893918	8.41	2.1294214	8.74	2.1679101
8.09	2.0906287	8.42	2.1306098	8.75	2.1690536
8.10	2.0918640	8.43	2.1317967	8.76	2.1701959
8.11	2.0930984	8.44	2.1329822	8.77	2.1713367
8.12	2.0943306	8.45	2.1341664	8.78	2.1724763
8.13	2.0955613	8.46	2.1353491	8.79	2.1736146
8.14	2.0967905	8.47	2.1365304	8.80	2.1747517
8.15	2.0980182	8.48	2.1377104	8.81	2.1758874
8.16	2.0992444	8.49	2.1388889	8.82	2.1770218
8.17	2.1004691	8.50	2.1400661	8.83	2.1781550
8.18	2.1016923	8.51	2.1412419	8.84	2.1792868
8.19	2.1029140	8.52	2.1424163	8.85	2.1804174
8.20	2.1041341	8.53	2.1435893	8.86	2.1815467
8.21	2.1053529	8.54	2.1447609	8.87	2.1826747
8.22	2.1065702	8.55	2.1459312	8.88	2.1838015
8.23	2.1077861	8.56	2.1471001	8.89	2.1849270
8.24	2.1089998	8.57	2.1482676	8.90	2.1860512
8.25	2.1102128	8.58	2.1494339	8.91	2.1871742
8.26	2.1114243	8.59	2.1505987	8.92	2.1882959
8.27	2.1126343	8.60	2.1517622	8.93	2.1894163
8.28	2.1138428	8.61	2.1529243	8.94	2.1905355
8.29	2.1150499	8.62	2.1540851	8.95	2.1916535
8.30	2.1162555	8.63	2.1552445	8.96	2.1927702
8.31	2.1174596	8.64	2.1564026	8.97	2.1938856
8.32	2.1186622	8.65	2.1575593	8.98	2.1949998
8.33	2.1198634	8.66	2.1587147	8.99	2.1961128
8.34	2.1210632	8.67	2.1598687	9.00	2.1972245

N	Logarithm	N	Logarithm	N	Logarithm
9.01	2.1983350	9.34	2.2343062	9.67	2.2690282
9.02	2.1994443	9.35	2.2353763	9.68	2.2700618
9.03	2.2005523	9.36	2.2364452	9.69	2.2710944
9.04	2.2016591	9.37	2.2375130	9.70	2.2721258
9.05	2.2027647	9.38	2.2385797	9.71	2.2731562
9.06	2.2038691	9.39	2.2396452	9.72	2.2741856
9.07	2.2049722	9.40	2.2407096	9 73	2.2752138
9.08	2.2060741	9.41	2.2417729	9.74	2.2762411
9.09	2.2071748	9.42	2.2428350	9.75	2.2772673
9.10	2.2082744	9.43	2.2438960	9.76	2.2782924
9.11	2.2093727	9.44	2.2449559	9.77	2.2793165
9.12	2.2104697	9.45	2.2460147	9.78	2.2803395
9.13	2.2115656	9.46	2.2470723	9.79	2.2813614
9.14	2.2126603	9.47	2.2481288	9.80	2.2823823
9.15	2.2137538	9.48	2 2491843	9.81	2.2834022
9.16	2.2148461	9.49	2.2502386	9.82	2.2844211
9.17	2.2159372	9.50	2.2512917	9.83	2.2854389
9.18	2.2170272	9.51	2.2523438	9 84	2.2864556
9.19	2.2181160	9.52	2.2533948	9.85	2.2874714
9.20	2.2192034	9.53	2.2544446	9.86	2.2884861
9.21	2.2202898	9.54	2.2554934	9.87	2.2894998
9.22	2.2213750	9.55	2.2565411	9.88	2.2905124
9.23	2.2224590	9.56	2.2575877	9.89	2.2915241
9.24	2.2235418	9.57	2.2586332	9.90	2.2925347
9.25	2.2246235	9.58	2.2596776	9.91	2.2635443
9.26	2.2257040	9.59	2.2607209	9.92	2.2945529
9.27	2.2267833	9.60	2.2617631	9 93	2.2955604
9.28	2.2278615	9.61	2.2628042	9.94	2.2965670
9.29	2.2289385	9 62	2.2638442	9.95	2.2975725
9.30	2.2300144	9.63	2.2648832	9 96	2.2985770
9.31	2.2310890	9.64	2.2659211	9.97	2.2995806
9.32	2.2321626	9.65	2.2669579	9.98	2.3005831
9.33	2.2332350	9.66	2.2679936	9.99	2.3015846
9.34	2.2343062	9.67	2.2690282	10.00	2.3025851

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